Exam **Discrete Mathematic (resit), Tuesday January 28th 2014, 15:00h-18:00h**
Bachelor program Knowledge Engineering (Year 1)
GG90-92, A.0 (Turnzaal)

This exam consist of 8 problems, totalling 100 points, which you can solve in any order you like. Be sure to motivate all your answers with calculations/proofs/arguments.
Remember that $\mathbb{N} = \{1, 2, 3, \ldots\}$. Good luck!

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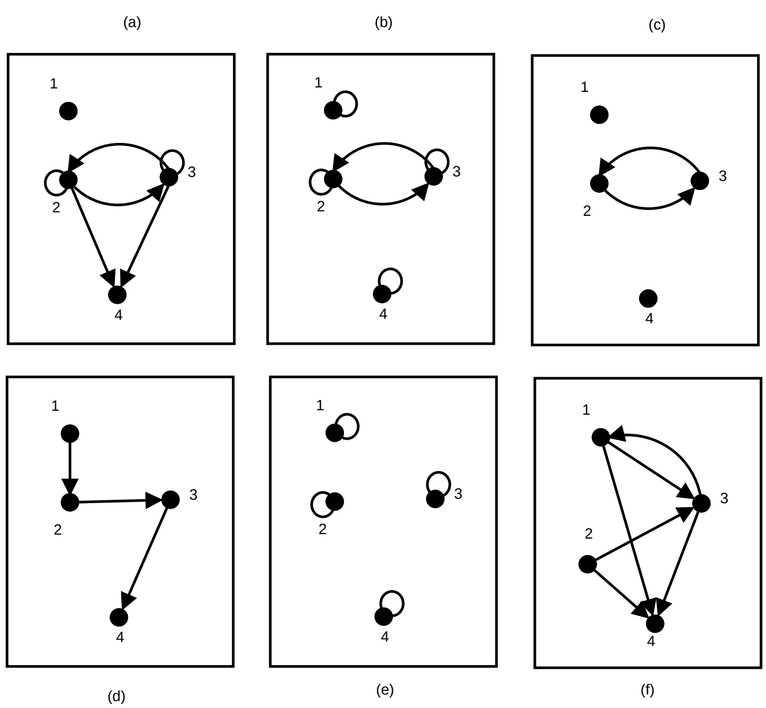
1. **(10 points)** Fill in the truth table for the following logical proposition:
   - $(p \wedge (q \Rightarrow \neg r)) \Leftrightarrow (p \vee r)$
2. **(15 points)** Use induction to prove the following statements:
   (a) For all integers $n \geq 0$,
   $$(3 \times 5^0) + (3 \times 5^1) + (3 \times 5^2) + \cdots + (3 \times 5^n) = \frac{3(5^{n+1} - 1)}{4}$$
   (b) For all integers $n \geq 1$, $4^{2n} - 1$ is divisible by 5.
3. **(15 points)** Prove or disprove, where the little "c" means "complement".
   - For all sets A, B, and C, $(B \subseteq C) \Rightarrow ((A \cup C^c) \subseteq (A \cup B^c))$.
4. **(15 points)** This question is about *relations*.
   (a) Let R be the relation on the set $\mathbb{R}$ defined as follows: xRy means "$|x - y| \geq 1$", where $|.|$ means "absolute value". Is R reflexive? Symmetric? Transitive? Anti-symmetric? For each of these properties, prove or disprove that it has that property. Is R an equivalence relation? Is R a partial order?
   (b) Let $A = \{1, 2, 3, 4\}$. For each of the six relations (a)-(f) shown below, state explicitly which of the properties it does and does not have: reflexive, symmetric, transitive, anti-symmetric.



5. **(10 points)** In class we discussed four different methods of selecting objects from a box. In the questions below, the different methods are illustrated by using the example of selecting 2 balls from a box containing 2 uniquely coloured balls: one (R)ed and one (G)reen.

   (a) **RG**. Calculate how many ways there are of (with this method) selecting 5 balls from a box containing 10 uniquely coloured balls.

   (b) **RG, GR**. Calculate how many ways there are of (with this method) selecting 4 balls from a box containing 8 uniquely coloured balls.

   (c) **RR, RG, GR, GG**. Calculate how many ways there are of (with this method) selecting 3 balls from a box containing 9 uniquely coloured balls.

   (d) **RR, RG, GG**. Calculate how many ways there are of (with this method) selecting 3 balls from a box containing 7 uniquely coloured balls.

6. **(10 points)** Prove or disprove the following statements.

   (a) $(\exists x \in \mathbb{Z})(\forall y \in \mathbb{N})(y + x \leq 7)$

   (b) $(\forall x \in \mathbb{N})(\exists y \in \mathbb{R})(xy = 1)$

   (c) $\neg(\forall x \in \mathbb{Z})(\forall y \in \mathbb{N})(xy < 0)$

   (d) $(\exists x \in \mathbb{N})(\forall y \in \mathbb{N})(\exists z \in \mathbb{Z})(y + z = x)$

7. **(15 points)** This is a question about *functions*.

   (a) For each of the statements below, explain (with motivation!) whether it is true or false.

      (1) If we define $f(x) = x^2$, and write $f : \mathbb{Z} \to \mathbb{Z}$, then this has the same injectivity/surjectivity properties as writing $f : \mathbb{R} \to \mathbb{R}$.

      (2) If $f : A \to B$ is injective then $|B| \geq |A|$. (You can assume here that A and B are finite sets).

      (3) If A contains 5 elements and B contains 4 elements, then all functions $f : A \to B$ are surjective.

   (b) Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined as follows: $f(x) = 2x^2$ (when $x < 0$), $f(x) = -4x$ (when $0 \leq x \leq 4$) and $f(x) = -\left(\frac{1}{x-4}\right) - 16$ (when $x > 4$). Is f invertible? If so, prove it. If not, explain why not. If f is invertible then define also the function f^{-1}.

8. **(10 points)** Let $A = \{1, 2, 3\}$, $B = \{5\}$, $C = \{2, 3, 5\}$.

   (a) Write down $(A \cup B) \setminus \emptyset$.

   (b) Write down $(A \setminus C) \times (A \cap C)$.

   (c) Write down $\mathbb{P}(B \times B)$.