Exam **Discrete Mathematics (resit), Tuesday 27th January 2015, 14:00h-17:00h**
Bachelor program Knowledge Engineering, year 1
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This exam consist of 8 problems, totalling 100 points, which you can solve in any order you like. Be sure to motivate all your answers with calculations/proofs/arguments.
Remember that $\mathbb{N} = \{1, 2, 3, \ldots\}$. Good luck!

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1. **(10 points)** Fill in the truth table for the following logical proposition.
   - $((p \Rightarrow (q \vee r))) \Leftrightarrow ((p \wedge \neg q) \Rightarrow r)$.
2. **(15 points)** Use induction to prove the following statements.
   (a) For all integers $n \geq 1$,
   $$\sum_{i=1}^{n} i(i+2) = \frac{n(n+1)(2n+7)}{6}$$
   (b) For all integers $n \geq 1$, $5^{2n} - 4$ is divisible by 3.
3. **(15 points)** Prove or disprove the following statements.
   (a) For all sets A, B and C, $(B \cap C^c) \cup (A \setminus B) = (A \cup B) \cap C^c$.
   (b) For all sets A, B and C, $(A \setminus B) \cup (C \setminus B) = (A \cup C) \setminus B$.
4. **(15 points)** This question is about *relations.*
   (a) Let R be the relation on $\mathbb{Z}$ defined as follows: xRy means "$x > y - \frac{1}{2}$". Is R reflexive? Symmetric? Transitive? Anti-symmetric? For each of these properties, prove or disprove that it has that property.
   (b) Let R be a relation on $\mathbb{Z} \setminus \{0\}$ defined as follows: xRy means"$xy \geq 1$". Prove or disprove that R is an equivalence relation. If it is an equivalence relation, describe its equivalence classes.
   (c) Let A be the set $\{0, 1\}$. Draw a relation diagram for a single relation R on A that *is* symmetric, transitive and anti-symmetric but which *is not* reflexive, or argue why such a relation cannot exist. (Note that the relation R does not have to have a "real-world" meaning).
5. **(15 points)** All the following questions are about *counting.*
   (a) Write down the inclusion-exclusion formula for two sets A, B.
   (b) 3 headache drugs - A,B, and C - were tested on 40 subjects. The results of tests:
   - 23 reported relief from (at least) drug A;
   - 18 reported relief from (at least) drug B;
   - 31 reported relief from (at least) drug C;
   - 11 reported relief from (at least) both drugs A and B;
   - 19 reported relief from (at least) both drugs A and C;
   - 14 reported relief from (at least) both drugs B and C;
   - 37 reported relief from at least one of the drugs.

   How many people got relief from *all 3* drugs? How many people got relief from *exactly 2* drugs?
   (c) How many different length 12 strings can you make using exactly 8 "0" symbols and exactly 4 "1" symbols? (Two strings are considered equal if and only they are identical in every position).
   (d) You roll 4 indistinguishable dice at the same time. How many different patterns are possible, assuming order is not important? (So, for example, *"Two sixes, one five, one three"* is considered a single pattern).

6. **(10 points)** Prove or disprove the following statements.

   (a) $(\forall x \in \mathbb{Z})(\forall y \in \mathbb{Z})(\exists z \in \mathbb{N})(x < z < y)$

   (b) $(\exists x \in \mathbb{Z})(\forall y \in \mathbb{N})(2x + 4y = 32)$

   (c) $(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})(x - y \geq 1)$

   (d) $\neg((\exists x \in \mathbb{N})(\exists y \in \mathbb{N})(xy = -1))$

7. **(15 points)** This is a question about *functions*.

   (a) Let S be the set $\{gold, silver, bronze\}$.

      (1) How many functions are there from S to S?

      (2) How many bijective functions are there from S to S?

      (3) Let us say that a function is *damaged* if it is *neither* an injection *nor* a surjection. How many damaged functions are there from S to S?

   (b) Let $\mathbb{R}^+ = \{x \in \mathbb{R} : x \geq 0\}$ and let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be the function defined as follows:

$$f(x) = \begin{cases} 2x^2 & \text{if } 0 \leq x < 3 \\ 4x + c & \text{if } x \geq 3 \end{cases}$$

   There exists only one possible value for the constant c which can make f invertible. What is it? Motivate this by proving that the function, for your choice of c, is invertible and give also f^{-1}.

8. **(5 points)** Let $A = \{1, \{2, 3\}, \emptyset\}$. Recall that $\mathbb{P}(A)$ denotes the *powerset* of A.

   (a) Write down $A \setminus \{1, 2\}$.

   (b) How many elements are there in $\mathbb{P}(A \setminus \emptyset)$? Explain briefly. (You do not need to list all the elements).