Exam **Discrete Mathematics, Thursday October 24th 2013, 09:00h-12:00h**
Bachelor program Knowledge Engineering
Tongersestraat 53, H 0.001 (Aula)

This exam consist of 8 problems, totalling 100 points, which you can solve in any order you like. Be sure to motivate all your answers with calculations/proofs/arguments.
Remember that $\mathbb{N} = \{1, 2, 3, \ldots\}$. **Good luck!**

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1. **(10 points)** Fill in the truth table for the following logical proposition:
   - $((p \Leftrightarrow \neg q) \vee r) \Rightarrow (p \wedge r)$
2. **(15 points)** Use induction to prove the following statements:
   (a) For all integers $n \geq 1$,
   $$\sum_{i=1}^{n} i(i+1) = \frac{n(n+1)(n+2)}{3}$$
   (b) For all integers $n \geq 1$, $11^n - 2^n$ is divisible by 9.
3. **(15 points)** Prove or disprove:
   - For all sets A, B, and C, $A \subseteq B \Rightarrow ((B \setminus A) \cap C) \cup (A \cap C) = C \cap B$.
4. **(15 points)** This question is about *relations*.
   (a) Let R be the relation on the set $\mathbb{Z}$ defined as follows: xRy means "$x \leq y + 1$". Is R reflexive? Symmetric? Transitive? Anti-symmetric? For each of these properties, prove or disprove that it has that property. Is R an equivalence relation? Is R a partial order?
   (b) Let $A = \{0, 1\}$. There are $2^3 = 8$ different ways of combining the properties reflexive, symmetric and transitive in a relation. For example, a relation might have all three properties, or none of them, or some but not others. For each of these 8 possible combinations, construct a relation R on A that has these properties and draw its relation diagram, or motivate why such a relation cannot exist. Note that A only contains two elements!
5. **(10 points)** At the start of each question below, there is a box with 12 balls in it, numbered 1 to 12 (i.e. each number appears exactly once).
   (a) A woman selects 5 balls from the box. After selecting a ball, she puts it back in the box. How many different selections can she make, assuming order *is not* important?
   (b) A child selects 6 balls from the box. After selecting a ball, she discards it. How many different selections can she make, assuming order *is* important?
   (c) A man selects 4 balls from the box. After selecting a ball, he puts it back in the box. How many different selections can he make, assuming order *is* important?
   (d) An alien selects 7 balls from the box. After selecting a ball, he discards it. How many different selections can he make, assuming order *is not* important?
6. **(10 points)** Prove or disprove the following statements.
   (a) $(\forall x \in \mathbb{Z})(\exists y \in \mathbb{N})(x + y \geq 7)$
   (b) $(\exists x \in \mathbb{N})(\forall y \in \mathbb{Z})(x + y \geq 7)$
   (c) $\neg(\forall x \in \mathbb{Z})(\exists y \in \mathbb{Z})(xy < 0)$
   (d) $(\forall x \in \mathbb{Z})(\exists y \in \mathbb{Z})(\forall z \in \mathbb{N})(x + y < z)$

7. **(15 points)** This is a question about *functions*.

   (a) For each of the statements below, explain (with motivation!) whether it is true or false. Assume that A and B are finite sets.

      (1) There are $|A|^{|B|}$ functions $f : A \to B$.

      (2) $f : A \to B$ is surjective if and only if $|B| \leq |A|$.

      (3) If $f : A \to B$ is bijective then $|A| = |B|$.

   (b) Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined as follows: $f(x) = 2x$ (when $x < 0$), $f(x) = x^2$ (when $0 \leq x \leq 3$) and $f(x) = \frac{1}{x-3} + 9$ (when $x > 3$). Is f invertible? If so, prove it. If not, explain why not. If f is invertible then define also the function f^{-1}.

8. **(10 points)** Let $A = \{1, 2, 3\}$, $B = \{1, 2\}$, $C = \{1, 3, 5\}$.

   (a) Write down $(A \setminus B) \times (C \setminus B)$.

   (b) Write down $\mathbb{P}(B \setminus A) \times C$

   (c) Write down all partitions of C.