Exam **Discrete Mathematics, Wednesday October 21st 2015, 09:00h-12:00h**
Bachelor program Knowledge Engineering, year 1
Tapijnkazerne + 0.015

This exam consist of 8 problems, totalling 100 points, which you can solve in any order you like. Be sure to motivate all your answers with calculations/proofs/arguments.
Remember that $\mathbb{N} = \{1, 2, 3, \ldots\}$**. Good luck!**

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1. **(10 points)** Fill in the truth table for the following logical proposition.
   - $((r \vee p) \Rightarrow \neg q) \wedge (p \Leftrightarrow (q \wedge r))$
2. **(15 points)** Use induction to prove the following statements.
   (a) For all integers $n \geq 1$,
   $$\sum_{i=1}^{n} (2i-1)^2 = \frac{n(2n-1)(2n+1)}{3}$$
   (b) For all integers $n \geq 1$, $7^n - 2^n$ is divisible by 5.
3. **(15 points)** Prove or disprove the following statements. (Recall that A^c denotes the *complement* of A and $\mathbb{P}(A)$ the *powerset* of A).
   (a) For all sets A, B and C, $(B \subseteq A^c \cup C) \Leftrightarrow (((A \cap B) \setminus (A \cap C)) = \emptyset)$.
   (b) For all sets A and B, $(\mathbb{P}(A) \cap \mathbb{P}(B) \neq \emptyset) \Rightarrow (A \cap B \neq \emptyset)$.
4. **(15 points)** This question is about *relations*.
   (a) Let R be the relation on $\mathbb{Z}$ defined as follows: xRy means "$\max(x, y) \leq x$", where $\max(x, y)$ denotes the maximum of the two numbers x and y. Is R reflexive? Symmetric? Transitive? Anti-symmetric? For each of these properties, prove or disprove that it has that property.
   (b) Let $A = \{0, 1, 2\}$. There are $2^3 = 8$ different ways of combining the properties *reflexive*, **anti-symmetric** and *transitive* in a relation. For example, a relation might have all three properties, or none of them, or some but not others. For each of these 8 possible combinations, construct a relation R on A that has these properties and draw its relation diagram, or motivate (very briefly) why such a relation cannot exist. Note that A only contains three elements. Note also that the relations you draw do not have to have a "real-world" meaning.
5. **(15 points)** All the following questions are about *counting*.
   (a) You roll 4 indistinguishable **8-sided** dice at the same time. How many different patterns are possible, assuming order is not important? (So, for example, *"Two sixes, one five, one eight"* is considered a single pattern.)
   (b) There are 150 students in a class. There are 3 bands (A)ha, (B)auhaus and (C)hvrches. 20 students do not like any of the bands, while 25 students like all three bands. 60 students like (A)ha and (C)hvrches (and possibly other bands too). 50 students like (C)hvrches and (B)auhaus (and possibly other bands too). Finally, 35 students like (A)ha and (B)auhaus (and possibly other bands too). How many students like *exactly one* band?
   (c) Let A and B be sets such that $|A| = 4$ and $|B| = 8$. How many injective functions are there from A to B?
   (d) How many different length 12 binary strings can you make using "0" and "1" symbols? (Two strings are considered identical if and only if they are identical in every position. Note that it is fine if a string uses only one type of symbol i.e. 000000000000 and 111111111111 both count).

6. **(10 points)** Prove or disprove the following statements.

   (a) $(\exists x \in \mathbb{R})(\forall y \in \mathbb{Z})(y + x \notin \mathbb{N})$

   (b) $\neg((\exists x \in \mathbb{Z})(\forall y \in \mathbb{N})(xy = 0))$

   (c) $(\forall x \in \mathbb{Z})(\exists y \in \mathbb{N})(x - y \leq -10)$

   (d) $(\forall x \in \mathbb{N})(\exists y \in \mathbb{Z})(\forall z \in \mathbb{Z})(x + y \leq z)$.

7. **(15 points)** This is a question about *functions*.

   (a) Prove or disprove: if $f : A \to B$ is *not* injective and $|A| \leq |B|$, then f is *not* bijective. (You can assume that A and B are finite sets.)

   (b) Suppose $A = \{0, 1, 2\}$ and $B = \{0, 1, 2, 3, 4\}$. Let $g : B \to B$ be defined as $g(x) = 4 - x$. Draw a function $f : A \to B$ such that $(g \circ f)(0) = 4$, $(g \circ f)(1) = 2$ and $(g \circ f)(2) = 0$. Recall that $(g \circ f)(x)$ is notationally equivalent to $g(f(x))$.

   (c) True or false: if $C = \{north, south, east, west\}$ and $D = \{0, 1, 2\}$, then there are $3^4 = 81$ possible functions $f : D \to C$. (No motivation required here).

   (d) Let $\mathbb{R}^+ = \{x \in \mathbb{R} : x \geq 0\}$ and let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be the function defined as follows, where ${}^+\sqrt{x}$ refers to the positive square root of x:

$$f(x) = \begin{cases} x^2 & \text{if } 0 \leq x \leq 6 \\ {}^+\sqrt{(x+10)} + c & \text{if } x > 6 \end{cases}$$

   There exists only one possible value for the constant c which can make f invertible. What is it? Motivate this by proving that the function, for your choice of c, is invertible and give also f^{-1}.

8. **(5 points)** Let $A = \{1, 2\}$, $B = \{1, \{2\}, \emptyset\}$.

   (a) Write down $A \times B$.

   (b) Write down $\mathbb{P}(A) \cap B$.