Intro to systems of linear equations

- Equivalent: if two systems have the same solution set
- Consistent: if a system has one solution or infinite
- Solving:
  - Add a (multiple) of a row
  - Switch the position of two rows
  - Multiple a row by non zero constant
- Existence: if there is a contradiction in the augmented matrix then there is no solution.
- Uniqueness: if there are slack variables, solution is not unique
- Row echelon form
  - All zero entries at the bottom
  - Leading entry is to the right of the row above it
  - All entries below a leading entry are zeros
- Reduced
  - The leading entry is 1
  - The leading entry is the only non-0
- A matrix has a unique reduced echelon matrix
- Pivot positions and columns are those positions that will correspond to 1s in the reduced row echelon form
- A homogeneous equation has a non-trivial solution if and only if the equation has at least one free variable
- A homogeneous linear system solution is usually expressed as a span, {0} if empty, {u} if a line, {u,v} if a plane, etc.

Linear Independence and Span

- **Characterization of Linearly Dependent Sets**

  An indexed set $S = \{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors in S is a linear combination of the others. In fact, if S is linearly dependent and $\mathbf{v}_1 \neq \mathbf{0}$, then some $\mathbf{v}_j$ (with $j > 1$) is a linear combination of the preceding vectors, $\mathbf{v}_1, \dots, \mathbf{v}_{j-1}$.
- If a set contains more vectors than there are entries in each vector, then the set is linearly dependent. That is, any set $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is linearly dependent if $p > n$.
- Span: the endpoints reached by a vector or set of vectors
- Linear independence: the vectors aren't on the same line, the new vector doesn't help the other reach more points

Inverse

- Find it my putting the identity matrix next to it, and doing row operations on the identity to get A, applying the same steps to A, A will now be the inverse
- Finding the inverse is interesting because then solving the SLE is nice by $A^{-1} \times b$
- A matrix is invertible if
  - Row equivalent to the identity matrix
  - No free variables
  - Pivots in every row/column
  - Linearly independent
  - Columns span R^n
  - $Ax = 0$ has the trivial solution only
  - $Ax = b$has only one solution
  - $det(A) \neq 0$
- We call non-invertible matrices **"singular"**

Determinant

- What is the determinant? What does it represent?
  - How much a transformation scales areas
- How do we find the determinant?
  -

Vector Spaces

- **Null space:**
  - Solution to $Ax = 0$
  - It is a subspace
  - Could be
    - 0 if the equation only has the trivial solution
    - Infinitely many solutions, that span the free variables

example:

$$A = \begin{bmatrix} 1 & 2 & 0 & -1 & 3 \\ 3 & 8 & 2 & 1 & 4 \\ 0 & 2 & 1 & 3 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & -1 & 3 \\ 0 & 2 & 2 & 4 & -5 \\ 0 & 0 & -1 & -1 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -3 & 20 \\ 0 & 1 & 0 & 1 & -8\frac{1}{2} \\ 0 & 0 & 1 & 1 & -6 \end{bmatrix}$$

So
$$\begin{cases} x_1 = 3x_4 - 20x_5 \\ x_2 = -x_4 + 8\frac{1}{2} x_5 \\ x_3 = -x_4 + 6x_5 \\ x_4 \text{ is free} \\ x_5 \text{ is free} \end{cases} \qquad \underline{x} = x_4 \begin{bmatrix} 3 \\ -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -20 \\ 8\frac{1}{2} \\ 6 \\ 0 \\ 1 \end{bmatrix}$$

So $\text{Nul } A = \text{span}\left\{ \begin{bmatrix} 3 \\ -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -20 \\ 8\frac{1}{2} \\ 6 \\ 0 \\ 1 \end{bmatrix} \right\}$ and $\dim \text{Nul } A = 2$

- What does it mean?
  - The free variables we have and the number of them
  - It tells us whether the matrix is invertible, which in turn tells us if there is a way to do the anti-transformation
- **Column space**
  - The columns of original matrix that have pivots
  - Basically what are the basic variables and how many are there

$$\begin{bmatrix} 1 & 2 & 7 \\ -2 & 5 & 4 \\ -5 & 6 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 7 \\ 0 & 9 & 18 \\ 0 & 16 & 32 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 7 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \quad B_{Col\,A} = \left\{ \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} \right\}$$

  The dimension of the column space is called a **rank.**
- **Row space**
  - Same as column space but for rows
  - The rank of a 2d invertible matrix is all the columns
  - It is a subspace
- **Relationship between row/column space and null space**
  - Any linear combination of the rows/columns subspaces are orthogonal to vectors in the null space
- **Rank**
  - Dimension of column space
- **Basis of vector space**
  - The minimum amount of linearly independent vectors needed to span a space (identity vectors)

Standard basis $\mathcal{E}$

$\mathbb{R}^2$: $\mathcal{E} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

$\mathbb{R}^3$: $\mathcal{E} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

Translations

- This is about working with an object of a lower dimension in a higher dimension, and making it compatible example:

We want to translate $\vec{x}$:

In $\mathbb{R}^2$, $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \xrightarrow{\text{translation}} \vec{v} = \begin{bmatrix} x_1 + h \\ x_2 + k \end{bmatrix}$

Homogenous coordinates: $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix}$ in $\mathbb{R}^3$

The translation matrix is: $\begin{bmatrix} 1 & 0 & h \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix}$

Eigenvectors

- Vectors that maintain the same space after a transformation
  - In other words the direction of them doesn't change and the transformation was only a scalar multiplication
  - Excluding the zero vector as this is trivial and of course it never changes
  - The scalar multiple is called the **eigenvalue**
  - There are an infinite number of eigenvectors for each eigenvalue, the space they occupy is called the **eigenspace**
- How to check if a vector is an eigenvector?
  - Try to transform it, and if it doesn't (maintains the same span/direction) and only scales then it's an eigenvector
- How to check if a value is an eigenvalue?
  - Well, this means $Ax = Px$ where p is the eigenvalue
  - So rearranging $(A - p)x = 0$ where p is the identity matrix multiplied by the eigenvalue

example:

$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is $\lambda = 1$ an eigenvalue?

(if so, there exists a $\vec{x} \neq 0$ such that $A\vec{x} = 1 \cdot \vec{x}$)

$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $(A - 1 \cdot I) = \begin{bmatrix} 0-1 & 1 \\ 1 & 0-1 \end{bmatrix} \sim \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$

There exists a free variable, so $\lambda = 1$ is an eigenvalue of A.

- How to find the eigenvalues of a matrix?

To find all the eigenvalues of $A = (n \times n)$:

1. Generate the determinant of $(A - \lambda \cdot J)$
2. Constrain the determinant to be 0: $|A - \lambda J| =$
3. This gives a polynomial of degree n
   → this is a characteristic polynomial of A
4. When solving, it gives n roots. They are the eigenvalues.

example:

$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ there are 2 eigenvalues

$(A - \lambda J) = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix}$

1. $\det(A - \lambda \cdot J) = (2-\lambda)(2-\lambda) - 1 \cdot 1 = 4 - 4\lambda + \lambda^2 - 1$
   $= \lambda^2 - 4\lambda + 3$
2. Constrain to 0 : 3. $\lambda^2 - 4\lambda + 3 = 0$
   $(\lambda - 1)(\lambda - 3) = 0$
   $\lambda - 1 = 0 \vee \lambda - 3 = 0$
   4. $\lambda = 1 \vee \lambda = 3$

So the two eigenvalues are $\lambda = 1$ and $\lambda = 3$

- **Properties of Eigenvalues**
  - If 0 is an eigenvalue then the matrix is not invertible
  - The product of all eigenvalues is equal to the determinant
  - The sum of all eigenvalues- is the trace of the matrix
    - **Trace** is the sum of the entries on the main diagonal, regardless of the form
- **Row operations do change the eigenvalues!**

- **Characteristic equation (characteristic polynomial):** $det(A - \lambda J) = 0$

Diagonalizable

- Making A diagonalizable means finding a matrix D such that $A = PDP^{-1}$
- To compute D we find the eigenvalues of A and place them on the diagonal
- A is diagonalizable if
  - There are n distinct eigenvalues or
  - Sum of the dimensions of the eigenspaces is equal to n
- Diagonalizable is useful to compute the power of a matrix, A^k
  - $A^k = PD^kP^{-1}$

    Where D^k is $D^k = \begin{bmatrix} a^k & 0 & \cdots & 0 \\ 0 & b^k & & \vdots \\ 0 & \cdots & \ddots & z^k \end{bmatrix}$
- How does this work?
  - P is transforming the matrix into a different coordinate system, keeping the location the same

- D is transforming the location
- P-1 is taking the new location back to the original coordinate system

Dot Product

- Multiplying two vectors and getting a scalar

example:

$\mathbb{R}^2$: $\vec{x} = \begin{bmatrix}3\\1\end{bmatrix}$ $\vec{y} = \begin{bmatrix}5\\8\end{bmatrix}$ $\vec{x}\cdot\vec{y} = \begin{bmatrix}3 & 1\end{bmatrix}\begin{bmatrix}5\\8\end{bmatrix} = 15 + 8 = 23$

$$\begin{bmatrix}1\\2\end{bmatrix}\cdot\begin{bmatrix}3\\4\end{bmatrix} = 1\cdot3+2\cdot4$$

- **Norm**
  - Length of a vector
  - Can be found by taking the square root of the dot product of a vector with itself

example:

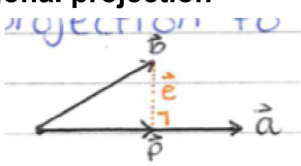
$\vec{x} = \begin{bmatrix}3\\2\end{bmatrix}$ $\|\vec{x}\| = \sqrt{\vec{x}\ \vec{x}} = \sqrt{\begin{bmatrix}3 & 2\end{bmatrix}\begin{bmatrix}3\\2\end{bmatrix}} = \sqrt{3^2+2^2} = \sqrt{15}$

- You can also find the distance between two vectors by taking the square root of the dot product of the distance between them

Orthogonal

- **Orthogonal vectors** means the two vectors are perpendicular to each other

  Two vectors are perpendicular when the dot product between them is equal to 0
- **Orthogonal subspaces (orthogonal complements)**, similarly, means that all the vector in a subspace are orthogonal to all the vectores in another subspace
- **Orthogonal projection**



  In this photo, vector b is projected onto a, vector p is a representation of that projection, finding the size of p is possible, it is in the span of a, so it is a linear combination of a (scaled down)

  It is possible to calculate the angle between two lines using this orthogonal projection
- **Orthogonal Matrix**

Orthogonal matrix: an $(n \times n)$ matrix with orthonormal columns such that $A^{-1} = A^T$

PROPERTIES:

1. $AA^T = A^TA = I$ so $A^T = A^{-1}$
2. $\|A\vec{x}\| = \|\vec{x}\|$ $\left(= \sqrt{(A\vec{x})^T \cdot (A\vec{x})} = \sqrt{\vec{x}^T A^T \cdot A \cdot \vec{x}} = \sqrt{\vec{x}^T \cdot \vec{x}} = \sqrt{\vec{x} \cdot \vec{x}} \right)$
3. $(A\vec{x}) \cdot (A\vec{y}) = \vec{x} \cdot \vec{y}$
4. $(A\vec{x}) \cdot (A\vec{y}) = 0$ iff $\vec{x} \perp \vec{y}$

The eigenvalues are:
- if complex: $\lambda = a + ib$ with $\sqrt{a^2+b^2} = 1$
- if real: $\lambda = 1$ or $\lambda = -1$

Symmetric matrix:

- $A = A^T$
- Always have n real eigenvalues
- Dimension of the orthogonal polynomial is equal to the multiplicity of the eigenvalue
- Can always be orthogonally diagonalized
- The eigenvectors from different eigenspaces are always perpendicular (orthogonal to each other)
- How do we find **orthogonal diagonalizable matrix?**
  - An orthogonally diagonalizable matrix is a subset of diagonalizable matrices
  - The difference is that orthogonal refers to ADA^{-1}, orthogonally diagonalizable is ADA^T

So to orthogonally diagonalize a matrix A we check:
- is A symmetric? → if not, stop.
- can we have n orthogonal eigenvectors?
  → if not, project one of the two not-orthogonal vectors $(\vec{u}_2)$ onto the other $(\vec{u}_1)$ and subtrac[t] this from $\vec{u}_2$
- is P an orthogonal matrix?
  → if not, divide every vector (column) by its no[rm]

Last notes

- Remember to check if the inverse is correct by multiplying $AA^{-1}$and making sure its equal to the identity
- Row space is the reduced one, column space is the original
- If still have time, learn about finding the determinant of a 4*4
- When finding the determinant always check if there are linear dependencies before doing it
- To compute the basis for a certain eigenvalue, do $A - \lambda I$then solve for nul (The subspace for an eigenvalue of 0 is equal to the subspace of nul)