Linear Algebra

Lecture 1

System of linear equations (SLE): $\begin{cases} x+y=20 \\ 2x+4y=74 \end{cases}$

Linear means no $x \cdot y$ or a factor above x or y, so no x^2 or $\sqrt{y}$.

In a function $y=mx+q$, q is the offset (intersection with y-axis) and m is the slope.

Slow view point = first point of view to linear equations. The geometrical interpretation is finding the intersection of lines in a plane ($\mathbb{R}^2$)

Basic algorithm to solve a SLE:
- Row reduction (/ Gaussian elimination)

Augmented matrix of a SLE: a system of linea equations summarized by a coëfficient matrix and a vector of righthand side numbers given in the matrix as well.

So, eg, $A = \begin{array}{cc} x & y \\ \end{array}$ $A=\begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix}$ $\underline{b} = \begin{bmatrix} 20 \\ 74 \end{bmatrix} \rightarrow [A:b] = \begin{bmatrix} 1 & 1 & \vdots & 20 \\ 2 & 4 & \vdots & 74 \end{bmatrix}$

Dimensions of a matrix:
(row × column) $\hat{=}$ number of equations × #variable

Equivalent matrices are matrices whose corresponding SLE's are all equivalent, meaning they share the same solution

Row equivalent: when one can be changed to the other (matrix) by a sequence of row operations. These are:
1. row replacement (① → ① - ②)
2. row scaling (② → ②/2)
3. row exchange (① → ② and ② → ①)

A SLE can have:

1) a unique solution $\}$
2) infinite many solutions $\}$ system is consistent
3) no solution $\}$ system is inconsistent

When we try to solve a SLE, the questions we ask ourself are:

- is there a solution?
- is the solution unique?

A matrix can be in three forms:

1) augmented matrix
2) echelon form
   - useful to see if there exists a solution and if it's unique.

   Conditions:
   - all non-zero rows are above zero-rows
   - pivot of a row is in a column to the right of the pivot of the above row
   - all entries below a pivot are zero

$$\left[\begin{array}{ccc:c} 2 & 1 & 3 & 4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 3 & 8 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

3) reduced row echelon form (RREF)

   Conditions:
   - pivots are equal to 1
   - each pivot is the only non-zero element in the column

$$\left[\begin{array}{ccc:c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

Pivot: left-most non-zero element in a row

Pivot columns define basic variables while ~~Pivot rows~~ other columns define free variables.

Going from a matrix to the echelon form

1. Begin with the leftmost non-zero column. This is a pivot column. The pivot position is at the top.
2. Select a non-zero entry in the pivot column as a pivot. If necessary, exchange rows to move this entry into the pivot position.
3. Use row replacement operations to create

zeroes in all positions below the pivot.
4. Cover the row containing the pivot position and all rows above it. Apply steps 1-3 to the matrix that remains and repeat until no more non-zero ~~elem~~ rows can be modifi

Going from echelon form to RREF:
1. Begin with the rightmost pivot and work upward to the left, creating zeroes abov each pivot using row replacement operatior If a pivot is not 1, make it 1 by using a scaling operation.

Example:

$$\begin{cases} 2x_2 - 8x_3 = 8 \\ x_1 - 2x_2 + x_3 = 0 \\ -4x_1 + 5x_2 + 9x_3 = -9 \end{cases} \rightarrow [A : \underline{b}] = \begin{array}{c c} \begin{array}{cccc} x_1 & x_2 & x_3 & \end{array} & \\ \left[\begin{array}{ccc:c} 0 & 2 & -8 & 8 \\ 1 & -2 & 1 & 0 \\ -4 & 5 & 9 & -9 \end{array}\right] & \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \end{array}$$

$$\sim \left[\begin{array}{ccc:c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right] \sim \left[\begin{array}{ccc:c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{array}\right] \sim \left[\begin{array}{ccc:c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 0 & 1 & 3 \end{array}\right] \sim$$

exchange ① and ② | replace ③ with ③+4×① | replace ③ with ③+1½×②

$$\left[\begin{array}{ccc:c} 1 & -2 & 0 & -3 \\ 0 & 2 & 0 & 32 \\ 0 & 0 & 1 & 3 \end{array}\right] \sim \left[\begin{array}{ccc:c} 1 & 0 & 0 & 29 \\ 0 & 2 & 0 & 32 \\ 0 & 0 & 1 & 3 \end{array}\right] \sim \left[\begin{array}{ccc:c} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

replace ② with ②-8×③, replace ① with ①-③ | ~~scale ② with ②/2~~ replace ① with ①+② | scale ② with ②/2

So the solution set in parametric form is:
$x_1 = 29$ $x_2 = 16$ $x_3 = 3$

Lecture 2

Matrix equation: $\boxed{A\underline{x} = \underline{b}}$ with x as the solutic vector.
Dimensions of A determine the amount of solutions
$A = (m \times n) \rightarrow$ m equations, n variables.
if $m > n$: - infinite many solutions
- no solution

Column-viewpoint: another way of solving a SLE, but by looking at the columns. Vector columns represent points in a plane.

Vector operations:

1. Scaling

$\underline{v}$ in $\mathbb{R}^n$ $= \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ $C \cdot \underline{v} = \begin{bmatrix} C \cdot v_1 \\ C \cdot v_2 \\ \vdots \\ C \cdot v_n \end{bmatrix}$

c in $\mathbb{R}^1 = c$

scalar multiple of v

2. Addition

$\underline{w}, \underline{v}$ in $\mathbb{R}^n$: $\begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ $\underline{w} + \underline{v} = \begin{bmatrix} w_1 + v_1 \\ w_2 + v_2 \\ \vdots \\ w_n + v_n \end{bmatrix}$

When using ~~any~~ both operations, it is called a linear combination of two vectors.

Vector equation: $\boxed{\underline{v} + \underline{w} = \underline{b}}$

If you can write a vector y as a linear combination ~~ro~~ of the vectors in a given set of p vectors in $\mathbb{R}^n$ $(\{v_1, v_2, \dots, v_p\})$, so:

$y = c_1 \cdot v_1 + c_2 \cdot v_2 + \dots + c_p \cdot v_p$, then y is a linear combination of $v_1 \dots v_p$ with weights $c_1 \dots c_p$.

Column point of view:

- any SLE can be written as a vector equation
- to solve a SLE, we need to find out if there exists a linear combination of the columns of A that results in $\underline{b}$.

No solutions

$\begin{cases} x + 2y = 3 \\ x + 2y = 4 \end{cases}$ (in row viewpoint: parallel lines)

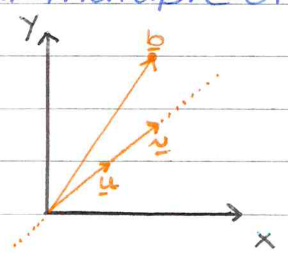
Column viewpoint: $A = \begin{array}{cc} u & v \\ \hline \end{array}$ $\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$ $\underline{b} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

take $\underline{u}, \underline{v}$ in $\mathbb{R}^2$

we see $\underline{v} = 2 \cdot \underline{u}$, so $\underline{v}$ is a scalar multiple of $\underline{u}$

There is no way we can linearly combine $\underline{u}$ and $\underline{v}$ and obtain $\underline{b}$

→ we cannot get off the line.



INFINITE MANY SOLUTIONS

$\begin{cases} x + 2y = 0 \\ 2x + 4y = 0 \end{cases}$ (Row viewpoint : same line)

Column viewpoint : $A = \begin{array}{cc} u & v \\ \end{array}\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ $\underline{b} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\underline{x} = \begin{bmatrix} x \\ y \end{bmatrix}$

take $\underline{u}, \underline{v}$ in $\mathbb{R}^2$

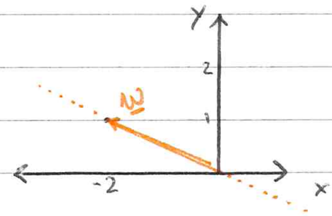
we can see that $\underline{v} = 2 \cdot \underline{u}$ and $\underline{b}$ is already on the line of $\underline{u}$ and $\underline{v}$, so there infinite many possibilities. Solution set : $\begin{cases} x = -2y \\ y \text{ is free} \end{cases}$

So : $\underline{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2y \\ y \end{bmatrix} = y \cdot \overset{\underline{w}}{\begin{bmatrix} -2 \\ 1 \end{bmatrix}}$

↳ parametric vector form

Geometrical interpretation

The solution set is a line in a plane (the line is defined by $\underline{w}$)



Span $\{\underline{w}\}$: the set of all possible scalar multiples of $\underline{w}$ (if $\{\underline{w}\}$ only contains one vector!)

To determine the span, we need to get to the parametric vector form:

1. Express the non-free variables in free variables
2. Fill $\underline{x}$ with only free variables and seperate them when there's more than one.
3. Write the vectors as multiples of a vector with the free-variable as scalar.

Example

$\{ x_1 - 3x_2 - 2x_3 = 0$ $\quad [A : b] = [1 \quad -3 \quad -2 \; \vdots \; 0]$

$\begin{cases} x_1 = 3x_2 + 2x_3 \\ 0 = 0 \\ 0 = 0 \end{cases} \rightarrow \begin{cases} x_1 = 3x_2 + 2x_3 \\ x_2 \text{ is free} \\ x_3 \text{ is free} \end{cases}$

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3x_2 + 2x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3x_2 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 2x_3 \\ 0 \\ x_3 \end{bmatrix} = x_2 \overset{\underline{v}}{\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}} + x_3 \overset{\underline{w}}{\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}}$$

so, $\underline{v}$ and $\underline{w}$ in $\mathbb{R}^3$

~~Thee~~ The span is $\{\underline{v}, \underline{w}\}$, because the solution set is any possible linear combination of two vectors in $\mathbb{R}^3$, thus a plane in $\mathbb{R}^3$.

The set of all possible linear combinations of the vectors in a given set of p vectors in $\mathbb{R}^n$; $\{v_1, v_2, \ldots, v_p\}$, is called $\text{span}\{v_1, v_2, \ldots, v_p\}$

→ Solving a SLE means to investigate if $\underline{b}$ is in the span of the columns of A.

→ if $\underline{b}$ is in the set of all possible linear combinations of the columns of A

Homogenous system (matrix view)

When the $\underline{b}$ vector is equal to zero, so:

$A \cdot \underline{x} = \underline{0}$ $(\underline{b} = \underline{0})$

$A = (m \times n)$ → $\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

It always has at least one solution!

→ Trivial solution : $x = 0$

In a homogenous system there are 2 possibilities:

1) Only trivial solution
   → no free variables, pivots in every column
2) Infinite many solutions
   → trivial solution + infinite others, free variables

> When solving a SLE where $b \neq 0$, the solution set is equal to that of a homogenous system plus a vector which is $\underline{b}$ modified by the same operations as those needed to get the homogenous system in RREF.

Example:

$\begin{cases} 2x + 4y = 6 \\ x + 2y = 3 \end{cases}$ $A = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}$ $\underline{b} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$ $\underline{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ $Ax = b$

1. $Ax = 0$

$[A \,\vdots\, 0] = \left[\begin{array}{cc:c} 2 & 4 & 0 \\ 1 & 2 & 0 \end{array}\right] \overset{②\leftrightarrow①}{\sim} \left[\begin{array}{cc:c} 1 & 2 & 0 \\ 2 & 4 & 0 \end{array}\right] \overset{②:②-2①}{\sim} \left[\begin{array}{cc:c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$

Solutionset: $\begin{cases} x = -2y \\ y \text{ is free} \end{cases}$ $\underline{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2y \\ y \end{bmatrix} = y \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

→ line in a plane

2. $Ax = b$

- $[A \vdots b] = \begin{bmatrix} 2 & 4 & \vdots & 6 \\ 1 & 2 & \vdots & 3 \end{bmatrix} \overset{②\leftrightarrow①}{\sim} \begin{bmatrix} 1 & 2 & \vdots & 3 \\ 2 & 4 & \vdots & 6 \end{bmatrix} \overset{②:②-2①}{\sim} \begin{bmatrix} 1 & 2 & \vdots & 3 \\ 0 & 0 & \vdots & 0 \end{bmatrix}$

- $\underline{b} = \begin{bmatrix} 6 \\ 3 \end{bmatrix} \underset{②\leftrightarrow①}{\sim} \begin{bmatrix} 3 \\ 6 \end{bmatrix} \underset{②:②-2①}{\sim} \begin{bmatrix} 3 \\ 0 \end{bmatrix}$

Solution set: $\begin{cases} x + 2y = 3 \\ 0 = 0 \end{cases} \rightarrow \begin{cases} x = 3 - 2y \\ y \text{ is free} \end{cases}$

$\underline{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 - 2y \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} + \begin{bmatrix} -2y \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} + y \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$=$ homogenous system $+ \begin{bmatrix} 3 \\ 0 \end{bmatrix}$

The solution set of $A\underline{x} = \underline{b}$ is a translated version of the solution set of $Ax = 0$, translated by a particular solution $\underline{p}$ of $A\underline{x} = \underline{b}$.

When you know the solution set of the homogenou[s] system, you can find the solution set of $Ax = b$ in 3 w[ays]

1. You solve $[A \vdots b] \sim$
2. You 'modify' $\underline{b} \sim$
3. You spot one particular solution of $Ax = b$ and ad[d] it to the solution set of $Ax = 0$.

Lecture 3

Linearly independent vectors

~~When~~ A set of p vectors is said to be linearly independent if the vector equation $c_1v_1 + \dots + c_pv_p = 0$ only has the trivial solution ($c = 0$).

So we cannot write a vector in the set as a linear combination of the other vectors in the set.

Remarks:

- If the set contains a 0-vector, it's automatically linearly dependent!
- If the set in $\mathbb{R}^n$ only contains one vector, for it to be linearly independent $\underline{v} \neq 0$.
- If the set in $\mathbb{R}^n$ only contains two vectors, for it to be linearly independent $\underline{v}_1$ and $\underline{v}_2$ may not be

scalar multiples of each other.
- if a set of vectors contains more vectors than entries in a vector, then the set is linearly dependent (if $p > n$ in $\{\underline{v}_1, \dots, \underline{v}_p\}$ in $\mathbb{R}^n$).
- maximum linearly independent vectors in $\mathbb{R}^n$ is n, because span $\{\underline{v}_1, \underline{v}_2, \dots, \underline{v}_{pn}\}$

lecture 4

Third point of view to solve a SLE:
The geometrical interpretation of the matrix equation $A\underline{x} = \underline{b}$.

OPERATIONS WITH MATRICES

- **Matrix vector multiplication:**

$A = (m \times n)$ $\quad \underline{b} = (n \times 1)$ $\quad$ product $= (m \times 1)$

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad \underline{b} = \begin{bmatrix} e \\ f \end{bmatrix}$ $\quad$ product $= \begin{bmatrix} a \cdot e + b \cdot f \\ c \cdot e + d \cdot f \end{bmatrix}$

properties:
- $A(\underline{u} + \underline{v}) = A \cdot \underline{u} + A\underline{v}$
- $A(c \cdot \underline{u}) = c \cdot A \cdot \underline{u}$ with c in $\mathbb{R}$

- **matrix-matrix addition**

only possible when both matrices have the same dimensions → sum is an element-wise sum

$A = \underset{(2 \times 3)}{\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}}$ $\quad B = \underset{(2 \times 3)}{\begin{bmatrix} u & v & w \\ x & y & z \end{bmatrix}}$ $\quad A + B = \underset{(2 \times 3)}{\begin{bmatrix} a+u & b+v & c+w \\ d+x & e+y & f+z \end{bmatrix}}$

- **scalar multiplication**

$c \cdot A$ with c in $\mathbb{R}$

$A = \underset{(2 \times 2)}{\begin{bmatrix} a & b \\ x & y \end{bmatrix}}$ $\quad c \cdot A = \underset{(2 \times 2)}{\begin{bmatrix} c \cdot a & c \cdot b \\ c \cdot x & c \cdot y \end{bmatrix}}$

- **matrix-matrix multiplication**

only possible when the number of columns of A are equal to the number of rows of B

$A = (m \times n)$ $\quad B = (n \times p)$ $\quad A \cdot B = (m \times p)$

$$A = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \quad B = \begin{bmatrix} r & s & t \\ u & v & w \\ x & y & z \end{bmatrix}$$

$$A \cdot B = \begin{bmatrix} a\cdot r + b\cdot u + c\cdot x & a\cdot s + b\cdot v + c\cdot y & a\cdot t + b\cdot w + c\cdot z \\ d\cdot r + e\cdot u + f\cdot x & d\cdot s + e\cdot v + f\cdot y & d\cdot t + e\cdot w + f\cdot z \end{bmatrix}$$

- matrix transpose (A^T)

the rows and columns have been swapped

$$A = \underset{(2\times 3)}{\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}} \qquad A^T = \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix} (3\times 2)$$

properties:

- $(A^T)^T = A$
- $(A+B)^T = A^T + B^T$
- $(r \cdot A)^T = r \cdot A^T$ with r in $\mathbb{R}^1$
- $(AB)^T = B^T \cdot A^T$ (reverse order)

- power of a matrix

only defined for square matrices ($n\times n$).

$$A^k = \underbrace{A \cdot A \cdot \ldots \cdot A}_{k}$$

Identity matrix A^0 (J_n)

Matrix filled with zeroes, except the main diagonal, which is filled with ones.

$$J_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

All identity matrices are symmetric matrices:

Symmetric matrix : its transpose is equal to the original matrix.

LINEAR TRANSFORMATIONS (equation $A\underline{x} = \underline{b}$)

Matrix A is seen as an operator that "transforms" $\underline{x}$ into $\underline{b}$, so A is a matrix transformation: a linear transformation that can be represented by a matrix.

$A = (m\times n)$ $\quad \underline{b} = (m\times 1)$ $\quad \underline{x} = (n \times 1)$

$$A : \mathbb{R}^n \rightarrow \mathbb{R}^m$$

$$x \xmapsto{A} \underline{b} = A \cdot \underline{x}$$

Solving a SLE ($A\underline{x} = \underline{b}$) means to find all vectors $\underline{x}$ in $\mathbb{R}^n$ (with $A = (m \times n)$) which are transformed into a vector $\underline{b}$ in $\mathbb{R}^m$, under the action of A.

TRANSFORMATION

a transformation (/mapping) from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns to each vector $\underline{x}$ in $\mathbb{R}^n$ a vector $\mathfrak{J}(x)$ in $\mathbb{R}^m$: $\mathfrak{J} : \underset{\text{domain}}{\mathbb{R}^n} \rightarrow \underset{\text{co-domain}}{\mathbb{R}^m}$

just a symbol

- $\mathfrak{J}(\underline{x})$ is the image of $\underline{x}$
- the set of all images $\mathfrak{J}(\underline{x})$ is the range of $\underline{x}$

a transformation is linear if :

1) $\mathfrak{J}(\underline{u} + \underline{v}) = \mathfrak{J}(\underline{u}) + \mathfrak{J}(\underline{v})$
2) $\mathfrak{J}(c \cdot \underline{v}) = c \cdot \mathfrak{J}(\underline{v})$ with c in $\mathbb{R}^1$

! all matrix transformations are linear transformations!

And! $\mathfrak{J}(\vec{0}) = \vec{0}$!

a matrix transformation is completely determined by what it does to the columns of the identity matrix.

example : we have a linear transformation $\mathfrak{J}(\underline{x})$ and we know $A = (3 \times 2)$. What is A?

answer: we must be able to do $A \cdot \underline{x}$ so:

$\underline{x} = (2 \times 1)$ and $\underline{b} = (3 \times 1)$

So: $\mathfrak{J}(\underline{x}) : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ (we want to get $\underline{b} = A \cdot x$)

take $\underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $\mathfrak{J}(\underline{x}) = A \cdot \underline{x} = \begin{bmatrix} k x_1 \\ k x_2 \\ k x_1 + h x_2 \end{bmatrix}$ take k, h in $\mathbb{R}^1$

because the domain is $\mathbb{R}^2 : J_2$

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
$e_1 \quad e_2$

$\mathfrak{J}(\underline{e}_1) = \mathfrak{J}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} k \\ 0 \\ k \end{bmatrix}$ $\mathfrak{J}(\underline{e}_2) = \mathfrak{J}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ k \\ h \end{bmatrix}$

$A = \left[\mathfrak{J}(\underline{e}_1) \quad \mathfrak{J}(\underline{e}_2)\right] = \begin{bmatrix} k & 0 \\ 0 & k \\ k & h \end{bmatrix}$

lecture 5

We can rotate shapes in $\mathbb{R}^2$ by applying the matrix transformation, that represents the rotation, to the vertices of the shape.

We can combine linear combinations, useful when rotating shapes:

1. translate the matrix (collecting the vertices) to 0.
2. rotate it (by means of matrix transformation)
3. translate the rotated matrix back

How to find a matrix E that gives the same result, when multiplied with A, as a row operation

- E acts on the rows of A → it's applied on the left.

1. Row reduce the matrix to RREF and write down the row operation.
2. Apply the same row operation (for every step separately) to the corresponding identity matrix. Now you obtained E_n, with n equal to the step
3. Multiply every E_n to obtain E.

→ E is the inverse of A.

INVERSE A^{-1}

The inverse multiplied with the original matrix gives ~~a~~ the identity matrix. A matrix is invertible if the RREF of A is I and every inverse is unique.

Properties:

If A is invertible matrix, then A^{-1} is also invertible:

1. $(A^{-1})^{-1} = A$
2. $(AB)^{-1} = B^{-1} \cdot A^{-1}$ (reverse order)
3. $(A^T)^{-1} = (A^{-1})^T$

How to compute the inverse?

$A : (n \times n)$ $\quad I_n \quad$ $[A \,\vdots\, I_n]$

Row reduce $[A \,\vdots\, I_n]$ until you get $[I_n \,\vdots\, A^{-1}]$ and you can read off A^{-1}

$$[A \,\vdots\, I_2] = \left[\begin{array}{cc:cc} 1 & 1 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc:cc} 1 & 1 & 1 & 0 \\ 0 & \cancel{2}1 & 0 & \frac{1}{2} \end{array}\right] \sim \left[\begin{array}{cc:cc} 1 & 0 & 1 & -\frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} \end{array}\right] \text{ so } A^{-1} = \begin{bmatrix} 1 & - \\ 0 & \end{bmatrix}$$

To solve $A\underline{x} = \underline{b}$ you can either:

1) row reduce $[A \vdots b] \sim$
2) if A is invertible, $\underline{x} = A^{-1} \cdot b$

Singular matrix: a non-invertible matrix

lecture 6

Determinant: number associated with a square matrix, it tells whether a matrix is invertible.
→ A is invertible if $\det(A) = |A| \neq 0$.

Invertible matrix theorem:
A is invertible if and only if:
- it's row equivalent to the identity matrix
- there's pivots in every row and column
- there are no free variables
- the columns of A are linearly independent
- columns of A span $\mathbb{R}^n$
- $A\underline{x} = \underline{0}$ has only the trivial solution $\underline{x} = \underline{0}$
- $A\underline{x} = \underline{b}$ has only one solution
- $\det(A) \neq 0$

How to compute the determinant:
1) Cofactor expansion
2) Gaussian elimination

Cofactor expansion
Given $A = (n \times n)$, focus on a certain row i or column j (preferably with the most zeroes).
$\det(A) = |A| = \sum_{j=0}^{n} a_{ij} C_{ij}$
with: a_{ij} is the entry of A at position (i,j)
C_{ij} is the cofactor $(i,j) = (-1)^{i+j} \cdot \det(A_{ij})$
A_{ij} is the submatrix given by removing row i and column j from A.

The determinant of a (2×2) matrix is equal to the product of the main diagonal minus the product of the other diagonal $(ad - bc)$

The determinant of a triangular or a diagonal matrix is equal to the product of the entrie on the main diagonal.

example: $A = \begin{bmatrix} 3 & 5 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ focus on column 1

$\det(A) = \sum_{i=1}^{3} a_{i1} C_{i1}$

$= 3 \cdot C_{11} + 0 \cdot C_{21} + 0 \cdot C_{31}$

$C_{11} = (-1)^{1+1} \cdot \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} = 1 \cdot (2 \cdot 1 - 1 \cdot 0) = 2$

So $\det(A) = 3 \cdot 2 = 6$

(or; upper triangular matrix so $\det(A) = 3 \cdot 2 \cdot 1 = 6$)

lecture 7

Gaussian elimination

We know the echelon form of a matrix is an upp triangular matrix, so we just need to know how row operations change the determinant and apply the opposite to the determinant to get the original:

* row replacement: no effect
* row scaling: multiply determinant with $\frac{1}{\text{scaling}}$
* row exchange: multiply determinant with -1

example:

$A = \begin{bmatrix} 0 & 5 & 1 \\ 4 & -3 & 0 \\ 2 & 4 & 1 \end{bmatrix} \overset{\boxed{1}}{\sim} \begin{bmatrix} 2 & 4 & 1 \\ 4 & -3 & 0 \\ 0 & 5 & 1 \end{bmatrix} \overset{\boxed{2}}{\sim} \begin{bmatrix} 2 & 4 & 1 \\ 0 & -11 & -2 \\ 0 & 5 & 1 \end{bmatrix} \overset{\boxed{3}}{\sim} \begin{bmatrix} 2 & 4 & 1 \\ 0 & 1 & 2/11 \\ 0 & 5 & 1 \end{bmatrix} \overset{\boxed{4}}{\sim} \begin{bmatrix} 2 & 4 & 1 \\ 0 & 1 & 2/11 \\ 0 & 0 & 1/11 \end{bmatrix}$

$\boxed{1}$ exchange ① and ②. D: $\times -1$

$\boxed{2}$ ②: ② $-2 \cdot$ ① D: no effect

$\boxed{3}$ ②: ② $/ -11$ D: $\times -11$

$\boxed{4}$. ③: ③ $-5 \cdot$ ② D: no effect

so $\det(A) = (2 \cdot 1 \cdot \frac{1}{11}) \cdot -1 \cdot -11 = 2$

Properties of determinant

1) $\det(A^T) = \det(A)$
2) if A and B are both square matrices: $\det(AB) = \det(A) \cdot \det$
3) if $A = (n \times n)$ and c in $\mathbb{R}$: $\det(c \cdot A) = c^n \cdot \det(A)$

Not recommendable!

Computing the inverse by means of cofactor expansio

if A is invertible, $A^{-1} = \frac{1}{|A|} \cdot \operatorname{adj} A$

adj A = adjugate matrix: matrix of a cofactors, transposed.

lecture 8

Vector spaces: a space populated by vectors.
To have a vector space, we need to be able to:
1). add vectors in the space and still stay in the space
2.) multiply a vector in the space by a scalar and stay in the space.
(1+2) linear combinations of vectors in the vector space gives vectors in the same space
3.) zerovector ($\underline{0}$) must be in the vector space.
→ When a condition is true, we call it close under the said 'rule'.

Vector spaces are linked to the definition of linear transformation:
1. $\mathcal{T}(\underline{u}+\underline{v}) = \mathcal{T}(\underline{u}) + \mathcal{T}(\underline{v})$
2. $\mathcal{T}(c \cdot \underline{v}) = c \cdot \mathcal{T}(\underline{v})$ with c in $\mathbb{R}$
3. $\mathcal{T}(\underline{0}) = \underline{0}$

In general, $\mathbb{R}^n$ is a vector space.
Polynomial spaces are also vector spaces.
↳ $\mathbb{P}^2: ax^2 + bx + c = 0$

Subspace: part of a vector space which is a vector space itself. Three properties to be fulfilled:
1) Close under vector addition
2) Close under scalar multiplication
3) $\underline{0}$ is in the subspace
→ because it's already in a vector space, all the other properties are fulfilled.

All possible subspaces of $\mathbb{R}^n$:
1. $\mathbb{R}^n$
2. all hyperplanes through the origin ($\underline{0}$)
3. all hyperlines through the origin ($\underline{0}$)
4. all hyperspaces through the origien ($\underline{0}$)
5. zerovector ($\underline{0}$)

We can define the dimension of a vector (sub)spa
by the required number of linearly independen
vectors to span a vector space.
In $\mathbb{R}^n$ we need n linearly independent vectors.
in $\mathbb{R}^2$ we can use span{$\underline{v}$} : a line through $\underline{0}$

In general, any span of p vectors in $\mathbb{R}^n$ is a subspa
of $\mathbb{R}^n$.
Dimensions: number of pivots in echelon form
number of columns –
Link between vector spaces and SLE's:
Two vector spaces connected to coëfficient
matrix A:
1. column space of A: Col A
2. Null space of A: Nul A

COLUMN SPACE

in echelon form
Col A = span of the pivot columns of A = span{$a_1, ..., a_n$}
To solve a SLE find out if $\underline{b}$ is in Col A ($A\underline{x} = \underline{b}$).
The column space is a subspace of $\mathbb{R}^m$
with m the number of rows in A.
Its dimensions are the number of pivots (in A).
example:

$$A = \begin{bmatrix} 1 & 1 \\ 4 & 2 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & -2 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

- columns of A are in $\mathbb{R}^3$
- Col A is strictly a subspac
- dim Col A = 2 (plane in $\mathbb{R}^3$)
- span{$\begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$}

NULL SPACE

A vector space that collects all the solutions to the
homogenous system $A\underline{x} = \underline{0}$. It contains all $\underline{x}$'s such
that $A\underline{x} = 0$ thus: Nul A = {x : x is in $\mathbb{R}^n$ and $A\underline{x} = 0$}
To find Nul A, we need to solve $A\underline{x} = 0$.
Two options:
1. $A\underline{x} = 0$ has only the trivial solution. ~~Null~~
   Nul A = ∅ and dim Nul A = 0
2. $A\underline{x} = 0$ has infinite many solutions (look at RREF
   Nul A is a subspace of $\mathbb{R}^n$.

Nul A = span{free variables}
dim Nul A = number of free variables

example:

$$A = \begin{bmatrix} 1 & 2 & 0 & -1 & 3 \\ 3 & 8 & 2 & 1 & 4 \\ 0 & 2 & 1 & 3 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & -1 & 3 \\ 0 & 2 & 2 & 4 & -5 \\ 0 & 0 & -1 & -1 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -3 & 20 \\ 0 & 1 & 0 & 1 & -8\tfrac{1}{2} \\ 0 & 0 & 1 & 1 & -6 \end{bmatrix}$$

so
$$\begin{cases} x_1 = 3x_4 - 20x_5 \\ x_2 = -x_4 + 8\tfrac{1}{2}x_5 \\ x_3 = -x_4 + 6x_5 \\ x_4 \text{ is free} \\ x_5 \text{ is free} \end{cases} \qquad \underline{x} = x_4 \begin{bmatrix} 3 \\ -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -20 \\ 8\tfrac{1}{2} \\ 6 \\ 0 \\ 1 \end{bmatrix}$$

So $\text{Nul } A = \text{span}\left\{ \begin{bmatrix} 3 \\ -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -20 \\ 8\tfrac{1}{2} \\ 6 \\ 0 \\ 1 \end{bmatrix} \right\}$ and dim Nul A = 2

lecture 9

Geometric interpretation of Nul A:
Nul A is the solution set of $A\underline{x} = \underline{0}$ where A is a matrix transformation that transforms any vector in the solution set of $A\underline{x} = \underline{0}$ into $\underline{0}$.

When $A = (n \times n)$, we can compute the determinant and in some cases the inverse.
Two cases:

1). A is square such that dim Nul A = 0, meaning $A\underline{x} = \underline{0}$ only has the trivial solution Nul A = {0} meaning there are no free variables so det A ≠ 0 so A is invertible. Thus, given any vector $\underline{x}$ in $\mathbb{R}^n$ such that $A\underline{x} = \underline{b}$, the transformation can be inverted: $A^{-1} \cdot \underline{b} = \underline{x}$

2) $A\underline{x} = \underline{0}$ has infinite many solutions so $0 < \text{dim Nul } A < n$ and Nul A ≠ {0} so there are free variables.
So det(A) = 0 so the matrix is not invertible so the ~~A~~ transformation cannot be reverted (we cannot get $\underline{x}$ back)

BASIS OF VECTOR SPACES

a basis for $\mathbb{R}^n$ is the minimal set of vectors in $\mathbb{R}^n$, meaning the minimum amount of ~~vectors~~ linearly independent vectors needed to span $\mathbb{R}^n$

Standard basis $\mathcal{E}$: columns of I_n:

$\mathbb{R}^2$: $\mathcal{E} = \{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\}$ $\qquad$ $\mathbb{R}^n$: $\mathcal{E} = \{[e_1 \dots e_n]\}$ $I_n = [e_1 \dots e_n]$

$\mathbb{R}^3$: $\mathcal{E} = \{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\}$ $\qquad$ etc.

We can define different coordinate systems, where each point is represented uniquely.
Points do not differ in location, only in the way their location is referred to.

The 'normal' coordinate system is under the standard basis: $[x]_\mathcal{E} = x$ $\rightarrow$ traditional coordinates

Changing the coordinate system can give a different representation of the same space, to highlight specific features.

To know the coordinates of a point relative to another basis (different from $[x]_\mathcal{E}$), we need to know the weights $c_1 \dots c_n$ such that $c_1 b_1 + \dots + c_n b_n = [x]_B$ with $x = \begin{bmatrix}c_1\\ \vdots \\ c_n\end{bmatrix}$

example:

$\vec{v}_1 = \begin{bmatrix}2\\1\end{bmatrix}$ $\quad$ $\vec{v}_2 = \begin{bmatrix}1\\2\end{bmatrix}$ $\quad$ $B = \{\vec{v}_1, \vec{v}_2\}$ is a basis for $\mathbb{R}^2$

express $(8,7)$ relative to B $\quad (= \vec{a})$

$\begin{bmatrix}8\\7\end{bmatrix} = [\vec{a}]_\mathcal{E} = 3 \cdot \vec{v}_1 + 2 \cdot \vec{v}_2$ $\quad$ so $[\vec{a}]_B = \begin{bmatrix}3\\2\end{bmatrix}$

The basis is a linear transformation that transform $[\vec{x}]_B$ into $[\vec{x}]_\mathcal{E}$. So: $\boxed{B \cdot [\vec{x}]_B = [\vec{x}]_\mathcal{E}}$

If B is invertibe, the transformation can also be reverted: $\boxed{[\vec{x}]_B = B^{-1} [\vec{x}]_\mathcal{E}}$

Basis for column space and null space:

NULL SPACE

The basis is the linearly independent vectors (or free variables) when written in the parametric vector form

example:

$$A(\text{RREF}) \sim \begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \underline{x} = \begin{bmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ 2x_5 - 2x_4 \\ x_4 \\ x_5 \end{bmatrix}$$

$$\underline{x} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

so $\dim \text{Nul } A = 3$

$$B_{\text{Nul } A} = \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

COLUMN SPACE

The basis is the pivot columns in the original matrix, so A has to be row reduced first.

$$\begin{bmatrix} 1 & 2 & 7 \\ -2 & 5 & 4 \\ -5 & 6 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 7 \\ 0 & 9 & 18 \\ 0 & 16 & 32 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 7 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \qquad B_{\text{Col } A} = \left\{ \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} \right\}$$

ROW SPACE

Row A is the set of all linear combinations of the rows of A.

dim Row A is the number of linearly independent rows.

$\dim \text{Row } A = \dim \text{Col } A^T = \dim \text{Col } A = \text{Rank}(A)$

Basis for ROW SPACE

The basis is the pivot rows in the EF or RREF matrices.

example:

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad B_{\text{Row } A} = \{ [1\ 0\ 0\ 2], [0\ 1\ 0\ 1] \}$$

RANK OF A MATRIX

In a matrix $A = (m \times n)$, n is equal to the number of free variables plus the number of basic variables, so:

$$n = \underbrace{\dim \operatorname{Nul} A}_{\text{free variables}} + \underbrace{\dim \operatorname{Col} A}_{\text{basic variables}} = \dim \operatorname{Nul} A + \operatorname{Rank}(A)$$

Rank of a matrix $= \operatorname{Rank}(A) = \dim \operatorname{Col} A$

where Rank tells you the number of ~~linearly independent~~ pivot columns.

If A is square, and invertible! $\operatorname{Rank}(A) = n$.

Four fundamental subspaces:

- $\operatorname{Col} A$
- $\operatorname{Nul} A$
- $\operatorname{Row} A$
- $\operatorname{Nul} A^T$

lecture 10

To translate an object we use homogenous coordinates. This means working in $\mathbb{R}^{n+1}$, with the last row of the to-be-translated object 1:

example:

We want to translate $\vec{x}$:

In $\mathbb{R}^2$, $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \xrightarrow{\text{translation}} \vec{v} = \begin{bmatrix} x_1 + h \\ x_2 + k \end{bmatrix}$

Homogenous coordinates: $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix}$ in $\mathbb{R}^3$

The translation matrix is: $\begin{bmatrix} 1 & 0 & h \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix}$

lecture 11

Eigenvectors of a square matrix $A = (n \times n)$ are the non-zero vectors for which the transformation $A\vec{x}$ lies in the same direction of $\vec{x}$. Meaning

the transformation $A\vec{x}$ is a scalar multiple of $\vec{x}$ where they're both on the same line.
So: $\boxed{A\vec{x} = \lambda\vec{x}}$ where $\vec{x} \neq 0$ and λ in $\mathbb{C}$
The scalar λ is called the eigenvalue.

For a homogenous system, if there are infinite many solutions, $\lambda = 0$ is an eigenvalue of A. This is also for non invertible matrices.

If A is square: $A = (n \times n)$
- There are always n eigenvalues, some of which might be complex
- If λ is k times an eigenvalue of A, then the "multiplicity" of λ is k

For each eigenvalue, there are infinite many eigenvectors and they all belong to a vector space, which is called the eigenspace of λ

To check if a vector $\vec{u}$ is an eigenvector of A, multiply A with $\vec{u}$ and see if $A\vec{u}$ is a scalar multiple of $\vec{u}$.
example:
$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $A\vec{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \neq \lambda \cdot \vec{u}$ so $\vec{u}$ is not an eigenvector!

To check if a scalar p is an eigenvalue of A, we need to check if there is a vector $\vec{x} \neq \vec{0}$ such that $\boxed{A\vec{x} = p\vec{x}}$.
So $A\vec{x} - p\vec{x} = \vec{0} \rightarrow \boxed{(A - p \cdot I)\vec{x} = \vec{0}}$.
So this homogenous system must have non-trivial solutions (to comply with $\vec{x} \neq \vec{0}$)
So $(A - p \cdot I)$ must have free variables for p to be an eigenvalue.
example:
$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is $\lambda = 1$ an eigenvalue?
(if so, there exists a $\vec{x} \neq 0$ such that $A\vec{x} = 1 \cdot \vec{x}$)

$J = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $(A - 1 \cdot J) = \begin{bmatrix} 0-1 & 1 \\ 1 & 0-1 \end{bmatrix} \sim \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$

There exists a free variable, so $\lambda = 1$ is an eigenvalue of A.

Eigenspace of $\lambda(=1)$ is the vectorspace collecting all eigenvectors of $\lambda(=1)$.
They are in the non-trivial solution set of $(A - \lambda \cdot J)\vec{x} = \vec{0}$ so:

Eigenspace of λ is equal to the Nullspace of $(A - \lambda \cdot J)$.

example:

$\lambda \stackrel{?}{=} 1$ $(A - 1 \cdot J) \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ $\begin{cases} x_1 = x_2 \\ x_2 \text{ is free} \end{cases}$ $\vec{x} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$Nul(A - 1 \cdot J) = span\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$ = eigenspace of $\lambda = 1$

So the basis for the eigenspace is equal to the basis for the $Nul(A - \lambda \cdot J)$

To find all the eigenvalues of $A = (n \times n)$:

1. Generate the determinant of $(A - \lambda \cdot J)$
2. Constrain the determinant to be 0: $|A - \lambda \cdot J| = 0$
3. This gives a polynomial of degree n
   → this is a characteristic polynomial of A
4. When solving, it gives n roots. They are the eigenvalues.

example:

$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ there are 2 eigenvalues

$(A - \lambda \cdot J) = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix}$

1. $\det(A - \lambda \cdot J) = (2-\lambda)(2-\lambda) - 1 \cdot 1 = 4 - 4\lambda + \lambda^2 - 1$
$= \lambda^2 - 4\lambda + 3$

2. Constrain to 0 : 3. $\lambda^2 - 4\lambda + 3 = 0$
$(\lambda - 1)(\lambda - 3) = 0$
$\lambda - 1 = 0 \vee \lambda - 3 = 0$
4. $\lambda = 1 \vee \lambda = 3$

So the two eigenvalues are $\lambda = 1$ and $\lambda = 3$

In a triangular matrix, the eigenvalues are the entries on the main diagonal.

PROPERTIES OF EIGENVALUES:

1. if $\lambda = 0$ is an eigenvalue of A, then A is not invertible
2. The product of all eigenvalues is equal to the determinant
3. The sum of all eigenvalues is equal to the trace (A) → the sum of the entries on the main diagonal, regardless of its form.
4. If there are n distinct eigenvalues, then you can find linearly independent eigenvectors.

Important: row operations do change the eigenvalues!

$\boxed{\det(A - \lambda \cdot I) = 0 \text{ is the characteristic equation}}$

lecture 12

MATRIX SIMILARITY

A matrix is similar to another matrix when there exists an ~~m~~ invertible matrix $P = (n \times n)$ such that:

$\boxed{A = PBP^{-1} \quad \Leftrightarrow \quad B = P^{-1}AP}$

If two matrices are ~~the same~~ similar, they have the same eigenvalues and the same determinant.

$\boxed{\det(P^{-1}) = \frac{1}{\det(P)}}$

MATRIX DIAGONALIZATION

A matrix is diagonalizable if we can find ~~an inv~~ a diagonal matrix D similar to it, such that: $A = P \cdot D \cdot P^{-1}$.

We can compute D by computing the eigenvalues of A and place them on the main diagonal of D.

We can compute P by computing the linearly independent eigenvectors of A, meaning that A must have distinct n eigenvalues.
To find the linearly independent vectors, you must calculate the eigenspaces for each eigenvalue.

If the eigenvalues are not all distinct, A can still be diagonalizable.
Given a specific eigenvalue of A, the number of linearly independent eigenvectors associated with its eigenspace is equal to the dimension of the eigenspace ($\dim \operatorname{Nul}(A-\lambda \cdot I)$)

In general, $\dim \operatorname{Nul}(A-\lambda \cdot I) \leq$ multiplicity of λ.
(If strictly $<$, then there are not enough linearly independent vectors.

The sum of dimensions of the eigenspaces has to be equal to n (= number of eigenvalues)

example 1:

$A = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$ A is triangular, so eigenvalues are entries on the main diagonal: $\lambda_1 = 1$ $\lambda_2 = 2$

so $D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

For P we need to find two linearly independent eigenvectors:

- $\lambda = 1$: $(A - 1 \cdot I) = \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $\vec{x} = \begin{cases} x_1 \text{ is free} \\ x_2 \text{ is } 0 \end{cases}$ $\vec{x} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$
- $\lambda = 2$: $(A - 2 \cdot I) = \begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$ $\vec{x} = \begin{cases} x_1 = 2x_2 \\ x_2 \text{ is free} \end{cases}$ $\vec{x} = x_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

So we can take $\vec{v}_1$ and $\vec{v}_2$ as our columns of P

$P = [\vec{v}_1 \ \vec{v}_2] = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ $P^{-1} = \frac{1}{\det(P)} \cdot \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{1} \cdot \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$

So: $PDP^{-1} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} = A$

example 2:

$A = \begin{bmatrix} -4 & 1 & 1 \\ 2 & -3 & 2 \\ 3 & 3 & -2 \end{bmatrix}$ is $\lambda = -5$ an eigenvalue? is A diagonalizable?

$(A - (-5) \cdot I) = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\vec{x} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$(A + 5I)\vec{x} = \vec{0}$

So the eigenspace of $\lambda = -5$ = Nul$(A + 5I)$

$= \text{span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$ dim = 2

therefore multiplicity of $\lambda = -5$ is 2

(there are at least 2 eigenvalues equal to -5)

if $\lambda_3 = -5$, then A is not diagonalizable.

Trace(A) = sum of entries on main diagonal

= sum of eigenvalues

$-4 - 3 - 2 = -5 - 5 + \lambda_3$

$\lambda_3 = 1 \quad \neq \lambda_1, \lambda_2$ so A is diagonalizable!

A is diagonalizable if:

- there are n distinct eigenvalues
- or the sum of the dimensions of the eigenspaces is equal to n.

To compute $A^k = \underbrace{A \cdot A \cdot \ldots \cdot A}_{k}$, we can also

diagonalize A and compute PD^kP^{-1}

! where D^k is equal to the entries on the main diagonal to the power of k:

$D^k = \begin{bmatrix} a^k & 0 & \cdots & 0 \\ 0 & b^k & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & z^k \end{bmatrix}$

lecture 13

Geometric interpretation of $A = PDP^{-1}$.

A is a matrix transformation, it represents a linear transformation $T(\vec{x})$:

$T(\vec{x}): \mathbb{R}^n \mapsto \mathbb{R}^n$ (A is square).

So $\vec{x} \mapsto A\vec{x} = \vec{b}$

Now, because A is diagonalizable:

$\vec{x} \mapsto A\vec{x}$

$\vec{x} \mapsto \underbrace{PDP^{-1}}\vec{x}$

sequence of 3 linear transformations:

1.) $P^{-1}\vec{x}$ P is invertible, so the columns of P are linearly independent, so there are n linearly independent vectors in $\mathbb{R}^n$ so the columns of P provide a basis for $\mathbb{R}^n$ → they also provide a representation of the P-coordinate system: $P^{-1}[\vec{x}]_\varepsilon = [\vec{x}]_P$ (location doesn't change)

2.) $D(P^{-1}[\vec{x}]_\varepsilon) = D[\vec{x}]_P = [\vec{b}]_P$
D transforms a point in $\mathbb{R}^n$ into another point in $\mathbb{R}^n$ (location does change)

3.) $P(DP^{-1}[\vec{x}]_\varepsilon) = P[\vec{b}]_P = [\vec{b}]_\varepsilon$
P transforms a point from the P-coordinate system to the standard coordinate system (ε) (location doesn't change)

Only D changes the location, so D is the equivalent of the transformation A in the P-coordinate system. So:

$\boxed{\text{D is the P-matrix of the transformation } \vec{x} \mapsto A\vec{x}.}$

This is true for any matrices A and C that are similar (C doesn't necessarily have to be diagonal):

$$\begin{array}{ccc} [\vec{x}]_\varepsilon & \xrightarrow{A} & [\vec{b}]_\varepsilon \\ P^{-1}\downarrow & & \uparrow P \\ [\vec{x}]_P & \xrightarrow[C]{} & [\vec{b}]_P \end{array}$$

COMPLEX EIGENVALUES

Two complex eigenvalues are a conjugate pair for a real matrix A (a matrix with only real entries) because the characteristic polynomial will have real coefficients.

When working with a rotation matrix, a complex number gives information about the angle of rotation (eigenvalues)
The eigenvectors give information about the direction of rotation.

$z = a + bi = \rho \cdot e^{i\cdot\theta} \longrightarrow \theta = atan(b/a)$

$z = i = 0 + 1 \cdot i \longrightarrow \theta = atan(b/0) = atan(1/0)$

with $\boxed{atan = tan^{-1} = arctan}$ $\theta = \pi/2$

Given a generic rotation of θ in $\mathbb{R}^2$:

$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ λ is complex for all θ such that $\theta \neq 2k\pi \vee \theta \neq (2k+1)\pi$

$C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ a and b are real and not both 0.

$\lambda_{1,2} = a \pm i \cdot b$

$z = a + bi = \rho \cdot e^{i\theta} = \rho(\cos\theta + i \cdot \sin\theta)$

$a = \rho \cos\theta$

$b = \rho \sin\theta$

So $C = \begin{bmatrix} \rho\cos\theta & -\rho\sin\theta \\ \rho\sin\theta & \rho\cos\theta \end{bmatrix} = \rho \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}_{\text{ROTATION MATRIX}}$

$= \underbrace{\begin{bmatrix} \rho & 0 \\ 0 & \rho \end{bmatrix}}_{\text{SCALING MATRIX}} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

So the linear transformation $\boxed{S : \vec{x} \mapsto C\vec{x} = SR\vec{x}}$

→ a sequence of two transformations hidden into C: a rotation and then a scaling

If A is a real (2x2) matrix, with a complex eigenvalue $\lambda = a - ib$ ($b \neq 0$) and an associated eigenvector $\vec{v}$ in $\mathbb{C}^2$, then we can always find a matrix C similar to A such that:

$A = PCP^{-1}$ $P = [Re\{\vec{v}\} \; Im\{\vec{v}\}]$

$C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ $\left[\vec{v} = \begin{bmatrix} i \\ 1 \end{bmatrix} \quad Re\{\vec{v}\} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad Im\{\vec{v}\} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right]$

Where C is a transformation equivalent to A in the P-coordinate system.
C is equivalent to a rotation followed by a scaling in the P-coordinate system.

lecture 14

INNER PRODUCT (DOT PRODUCT)

"Multiplying" two vectors $\vec{x}$ and $\vec{y}$ in $\mathbb{R}^n$ which results in a scalar ($\mathbb{R}$):

$$\vec{x} \cdot \vec{y} = \vec{x}^T \cdot \vec{y} \qquad (1 \times n)(n \times 1) = (1 \times 1)$$

example:

$$\mathbb{R}^2: \quad \vec{x} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \quad \vec{y} = \begin{bmatrix} 5 \\ 8 \end{bmatrix} \qquad \vec{x} \cdot \vec{y} = \begin{bmatrix} 3 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 8 \end{bmatrix} = 15 + 8 = 23$$

PROPERTIES:

1. $\vec{x} \cdot \vec{y} = \vec{y} \cdot \vec{x}$
2. $(\vec{x} + \vec{y}) \cdot \vec{w} = \vec{x} \cdot \vec{w} + \vec{y} \cdot \vec{w}$
3. $(c \cdot \vec{x}) \cdot \vec{y} = c \cdot (\vec{x} \cdot \vec{y})$ with c in $\mathbb{R}$
4. $\vec{x} \cdot \vec{x} \geq 0$, $= 0$ if $\vec{x} = 0$

Norm is the length of a vector, which is equal to the square root of the dot product of the vector with itself: $\|\vec{x}\| = \sqrt{\vec{x} \cdot \vec{x}}$ with $\vec{x}$ in $\mathbb{R}^n$
$= \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$

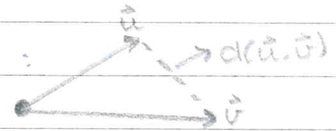
example:

$$\vec{x} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \qquad \|\vec{x}\| = \sqrt{\vec{x}\,\vec{x}} = \sqrt{\begin{bmatrix} 3 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix}} = \sqrt{3^2 + 2^2} = \sqrt{15}$$

PROPERTY

$\|c \cdot \vec{x}\| = |c| \cdot \|\vec{x}\|$ $\left(= \sqrt{(c \cdot \vec{x})(c \cdot \vec{x})} = \sqrt{c\vec{x}^T \cdot c\vec{x}} = \sqrt{c^2 \cdot \vec{x} \cdot \vec{x}} = \sqrt{c^2} \cdot \sqrt{\vec{x} \cdot \vec{x}} \right)$
(and $\|\vec{x}\|^2 = \vec{x} \cdot \vec{x}$)

Distance between two vectors:
$d(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\| = \|\vec{v} - \vec{u}\|$



UNIT VECTOR

A vector with length equal to 1, so $\|\vec{v}\| = 1$
To 'create' a unit vector from a vector, simply divide by the norm.

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \qquad \hat{v} = \frac{\vec{v}}{\|\vec{v}\|}$$

ORTHOGONAL VECTORS

Orthogonal vectors are vectors who are perpendicular to each other: $\vec{v} \perp \vec{w}$

Two vectors are perpendicular (orthogonal) if: $\vec{x} \cdot \vec{y} = 0$ or $\vec{y} \cdot \vec{x} = 0$

ORTHOGONAL SUBSPACES

Two subspaces are orthogonal if every vector in subspace A is orthogonal to every other vector in subspace B.

Therefore, A and B are both from the same subspace (otherwise $\vec{x} \cdot \vec{y}$ can't be computed)

The zero-vector ($\vec{0}$) is orthogonal to any other vector (and itself).

RELATIONSHIP BETWEEN COL A, ROW A, NUL A, NUL A^T

$A = (m \times n)$

- Row A is a subspace of $\mathbb{R}^n$
- Nul A is a subspace of $\mathbb{R}^n$

Any linear combination of the rows of A is orthogonal to $\vec{x}$, such that $\vec{x}$ is in Nul A.

So $\vec{x}$ is orthogonal to the span of the rows of A, so Row A.

So: $\boxed{\text{Row } A \perp \text{Nul } A}$

They are orthogonal complement, such that $(\text{Row } A)^{\perp} = \text{Nul } A$.

Because it's true for any matrix A (so also A^T):

$(\text{Row } A^T)^{\perp} = \text{Nul } A^T$

$(\text{Col } A)^{\perp} = \text{Nul } A^T$

To find an equation for Nul A (the line or plane it spans) you must multiply $\vec{x}$ with the rowvector (with pivot) of the RREF of A.

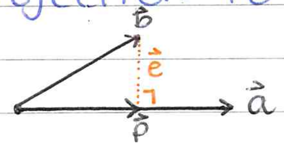
example:

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 4 & 10 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 5 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 + 2x_2 + 5x_3 = 0$$

$(A\vec{x} = 0)$

ORTHOGONAL PROJECTION

Projecting a vector $\vec{b}$ onto another vector $\vec{a}$ such that the end of the projection to the end of $\vec{b}$ is orthogonal to $\vec{a}$:



How can we find $\vec{p}$?

- $\vec{p}$ is in the span of $\{\vec{a}\}$, so $\vec{p} = x \cdot \vec{a}$ with x in $\mathbb{R}$.
- how can we find x?
  - $\vec{e} \perp \vec{a}$, with $\vec{e} = \vec{b} - \vec{p}$
  - so, $\vec{a} \cdot \vec{e} = 0 \rightarrow \vec{a}^T \cdot \vec{e} = 0$ (dot product)

$\vec{a}^T(\vec{b} - \vec{p}) = 0$ (substitution)

$\vec{a}^T(\vec{b} - x \cdot \vec{a}) = 0$ (substitution)

$\vec{a}^T\vec{b} - x \cdot \vec{a}^T \cdot \vec{a} = 0$ (working out brackets

$x = \frac{\vec{a}^T\vec{b}}{\vec{a}^T\vec{a}}$ (moving to other side $+ \div$

So $\vec{p} = x \cdot \vec{a} = \frac{\vec{a}^T\vec{b}}{\vec{a}^T\vec{a}} \cdot \vec{a} = PROJ_{\vec{a}}\,\vec{b}$

This is a linear transformation that transforms $\vec{b}$ into $\vec{p}$.

Corresponding matrix: $\boxed{P = \frac{\vec{a}\,\vec{a}^T}{\vec{a}^T\vec{a}}}$

(Because $\vec{p} = x\vec{a} = \vec{a} \cdot \frac{\vec{a}^T \cdot \vec{b}}{\vec{a}^T \cdot \vec{a}} = \frac{\vec{a}\,\vec{a}^T}{\vec{a}^T\vec{a}} \cdot \vec{b}$, so $P \cdot \vec{b} = \vec{p}$)

P is a projection matrix generated by 1 vector, so Rank (P) = 1 and P is symmetric $\rightarrow P^T = P$

We project to find an approximate solution to a SLE that is inconsistent.

This solution is called the Least Squares Solution: vector $\vec{x}$ is the orthogonal projection of the inconsistent vector onto the solution set (Col A) such that $\boxed{\vec{x} = (A^TA)^{-1}A^T \cdot \vec{b}}$ and the projection matrix is $P = I$.

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To find a line that best fits a set of data, use $y = ax + b$ and (x, y) of the points to create a SLE and find a vector that approximates the points as well as possible, so use LSS $((A^TA)^{-1}A^T \cdot b^{-1})$ to find $\vec{x}$ with $x_1 = a$ and $x_2 = b$ in $y = ax + b$.

example:

Points: (1,1), (2,2), (3,2) and $y = ax + b$

1. $\begin{cases} 1 = a + b \\ 2 = 2a + b \\ 2 = 3a + b \end{cases} \longrightarrow$ SLE: $\begin{cases} a + b = 1 \\ 2a + b = 2 \\ 3a + b = 2 \end{cases}$

2. So $A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix}$ $\quad \vec{b} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ $\quad \vec{x} = \begin{bmatrix} a \\ b \end{bmatrix}$

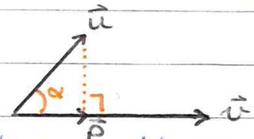
3. LSS: $(A^T A)\vec{x} = A^T\vec{b}$ (use RREF!)

$A^T A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 14 & 6 \\ 6 & 3 \end{bmatrix}$ $\quad A^T\vec{b} = \begin{bmatrix} 11 \\ 5 \end{bmatrix}$

4. So $\left[\begin{array}{cc:c} 14 & 6 & 11 \\ 6 & 3 & 5 \end{array}\right] \sim \left[\begin{array}{cc:c} 7 & 3 & 11/2 \\ 2 & 1 & 5/3 \end{array}\right] \sim \left[\begin{array}{cc:c} 1 & 0 & 1/2 \\ 0 & 3 & 2 \end{array}\right] \sim \left[\begin{array}{cc:c} 1 & 0 & 1/2 \\ 0 & 1 & 2/3 \end{array}\right]$

5. So $\vec{x} = \begin{bmatrix} 1/2 \\ 2/3 \end{bmatrix} \longrightarrow y = \frac{1}{2}x + \frac{2}{3}$

Angle between two vectors



To calculate the angle α, use the orthogonal projection of one vector onto the other.

So $\boxed{\cos\alpha = \dfrac{|\vec{u}^T \cdot \vec{v}|}{\|\vec{u}\| \cdot \|\vec{v}\|}}$ because $\vec{p} = \frac{\vec{u}^T\vec{v}}{\vec{v}^T\vec{v}} \cdot \vec{v}$

$\|\vec{p}\| = \left\|\frac{\vec{u}^T\vec{v}}{\vec{v}^T\vec{v}} \cdot \vec{v}\right\| = \left|\frac{\vec{u}^T\vec{v}}{\vec{v}^T\vec{v}}\right| \cdot \|\vec{v}\|$

$\|\vec{p}\| = \frac{|\vec{u}^T \cdot \vec{v}|}{\|\vec{v}\|^2} \cdot \|\vec{v}\| = \frac{|\vec{u}^T \cdot \vec{v}|}{\|\vec{v}\|}$ and $\frac{\|\vec{p}\|}{\|\vec{u}\|} = \cos\alpha$ $\left(\cos = \frac{a}{s}\right)$

ORTHOGONAL SET OF VECTORS

A set where all vectors are non-zero and orthogonal to each other, this is also a basis for the subspace (orthogonal basis).

To calculate the weights for a solution set of an orthogonal basis and a vector $\vec{x}$, instead of solving $A\vec{c} = \vec{x}$, we can use: $\boxed{c_i = \dfrac{\vec{x} \cdot \vec{v}_i}{\vec{v}_i \cdot \vec{v}_i} = \dfrac{\vec{x}^T \cdot \vec{v}_i}{\vec{v}_i^T \cdot \vec{v}_i}}$

ORTHONORMAL SET

Orthogonal set, with all vectors equal to length 1 (unit vectors)

Orthogonal set: $\{\vec{v}_1, \dots, \vec{v}_p\}$ dim $= p$
Orthonormal set: $\{\frac{\vec{v}_1}{\|\vec{v}_1\|}, \dots, \frac{\vec{v}_p}{\|\vec{v}_p\|}\}$ dim $= p$

The basis of an orthormal set is an orthonormal basis and it's a standard basis.

To go from a basis to an orthonormal basis: use Gram-Schmidt method:
$\{\vec{v}_1, \dots, \vec{v}_p\} \rightarrow \{\vec{e}_1, \dots, \vec{e}_p\}$ (standard basis)
$\hat{e}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$ $\quad \vec{e}_2 = v_2 - \text{PROJ}_{v_1} \vec{v}_2 \rightarrow \hat{e}_2 = \frac{\vec{e}_2}{\|\vec{e}_2\|}$

PROJECTING A VECTOR ONTO A SUBSPACE
To project a vector onto a subspace, create a matrix A such that its columns are the vectors in the basis.
To project the vector, multiply projection matrix P with the vector where
$P = A(A^T A)^{-1} A^T$.

If the basis is an orthonormal basis, then
$P = A\underbrace{(A^T A)}_{I}{}^{-1} A^T = A A^T$
because $\vec{a}_1 \cdot \vec{a}_2 = 0$ (if dim $H = 2$)
If A is a square matrix and invertible, then the columns of A are an orthonormal basis for subspace $H = \mathbb{R}^n$, so Col $A = \mathbb{R}^n$.
So there is no projection and $P = I$.

Orthogonal matrix: an $(n \times n)$ matrix with orthonormal columns such that $A^{-1} = A^T$

PROPERTIES:
1. $AA^T = A^T A = I$ so $A^T = A^{-1}$
2. $\|A\vec{x}\| = \|\vec{x}\|$ $\left(= \sqrt{(A\vec{x})^T \cdot (A\vec{x})} = \sqrt{\vec{x}^T A^T \cdot A \cdot \vec{x}} = \sqrt{\vec{x}^T \cdot \vec{x}} = \sqrt{\vec{x} \cdot \vec{x}}\right)$
3. $(A\vec{x}) \cdot (A\vec{y}) = \vec{x} \cdot \vec{y}$
4. $(A\vec{x}) \cdot (A\vec{y}) = 0$ iff $\vec{x} \perp \vec{y}$

The eigenvalues are:
- if complex: $\lambda = a + ib$ with $\sqrt{a^2 + b^2} = 1$
- if real: $\lambda = 1$ or $\lambda = -1$

DIAGONALIZATION OF SYMMETRIC MATRICES

Symmetric matrix is a square matrix such that $A = A^T$.

A is diagonalizable if the sum of the dimensions of the eigenspaces is equal to n, where $A = (n \times n)$

example:

$A = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$ $A^T = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$ $|A - \lambda I| = \begin{vmatrix} 2-\lambda & 3 \\ 3 & -6-\lambda \end{vmatrix} = 0$

$\rightarrow (2-\lambda)(-6-\lambda) - 9 = 0 \rightarrow \lambda^2 + 4\lambda - 21 = 0 \rightarrow (\lambda+7)(\lambda-3) = 0$

so $\lambda = -7$ and $\lambda = 3$ so A is diagonalizable with $D = \begin{bmatrix} -7 & 0 \\ 0 & 3 \end{bmatrix}$ and $P = [\vec{v}_1 \quad \vec{v}_2]$ so $A = PDP^{-1}$

$\lambda = 3$: $(A - 3 \cdot I)\vec{x} = \vec{0}$

$\begin{bmatrix} -1 & 3 \\ 3 & -9 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}$ $\vec{x} = x_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ so $\vec{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$\lambda = -7$: $(A + 7I)\vec{x} = \vec{0}$

$\begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1/3 \\ 0 & 0 \end{bmatrix}$ $\vec{x} = x_2 \begin{bmatrix} -1/3 \\ 1 \end{bmatrix}$ so $\vec{v}_2 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

So $P = \begin{bmatrix} -1 & 3 \\ 3 & 1 \end{bmatrix}$ and $\vec{v}_1 \cdot \vec{v}_2 = \vec{v}_1^T \cdot \vec{v}_2 = -3 + 3 = 0$ so $\vec{v}_1 \perp \vec{v}_2$

For a symmetric matrix, eigenvectors from different eigenspaces are always orthogonal to each other!

This does not mean that P is an orthogonal matrix! $P \cdot P^T = \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ we can make it orthogonal by dividing $\vec{v}_1$ and $\vec{v}_2$ by their norm.

So $\hat{P}\hat{P}^T = I$ and $A = \hat{P}D\hat{P}^T = \hat{P}D\hat{P}^{-1}$

So A is orthogonally diagonalizable.

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Symmetric matrices can always be orthogonally diagonalized.

So given any eigenvalue λ, the dimension of the eigenspace λ should equal the multiplicity of λ.

Eigenvectors from the same eigenspace

are (often) linearly independent but in general not orthogonal!
So to orthogonally diagonalize a matrix A we check:

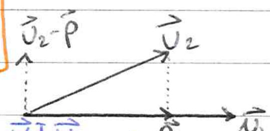
- is A symmetric? → if not, stop.
- can we have n orthogonal eigenvectors?
  → if not, project one of the two not-orthogonal vectors ($\vec{v}_2$) onto the other ($\vec{v}_1$) and subtrac this from $\vec{v}_2$
- is P an orthogonal matrix?
  → if not, divide every vector (column) by its no

example:

$$A = \begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix} \qquad A = A^T \checkmark$$

$$\underbrace{\lambda_1 = 7, \ \lambda_2 = 7}_{\text{multiplicity of 2}}, \ \lambda_3 = -2$$

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix} \qquad \vec{v}_3 = \begin{bmatrix} 1 \\ 1/2 \\ -1 \end{bmatrix}$$



$\vec{v}_1 \cdot \vec{v}_3 = 0$ ✓
$\vec{v}_2 \cdot \vec{v}_3 = 0$ ✓
$\vec{v}_1 \cdot \vec{v}_2 = -1/2$ ✗

make $\vec{v}_1 \perp \vec{v}_2$

$$z = \vec{v}_2 - \text{PROJ}_{\vec{v}_1} \vec{v}_2 = \vec{v}_2 - \frac{\vec{v}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \cdot \vec{v}_1$$

$$= \begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix} - \frac{-1/2}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/4 \\ 1 \\ 1/4 \end{bmatrix}$$

so $D = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & -2 \end{bmatrix}$ and $P = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 4 & 1/2 \\ 1 & 1 & -1 \end{bmatrix}$ (not orthogonal matrix yet)

with $\{\vec{v}_1, \vec{z}, \vec{v}_3\}$ as orthogonal set
and $\left\{\frac{\vec{v}_1}{\|\vec{v}_1\|}, \frac{\vec{z}}{\|\vec{z}\|}, \frac{\vec{v}_3}{\|\vec{v}_3\|}\right\}$ as orthonormal set
↳ which should be columns of $\hat{P}$

SPECTRAL THEOREM FOR SYMMETRIC MATRICES

1. Symmetric matrices always have n real eigenvalues
2. $\dim(A - \lambda I)$ = multiplicity of λ
3. eigenspaces are mutually orthogonal (eigenvectors from different eigenspaces are orthogonal
4. A is always orthogonally diagonalizable.