Calculus 1

December 17, 2008, 09:00-12:00

Clearly indicate how you came to your solution/conclusion.

Good luck with the exercises!

1. **(20 points)** Calculate

   a $\lim_{x\to -3} \frac{2x^2+4x-6}{x^2+x-6}$

   b $\lim_{x\to 4} \frac{|x^3-2x^2-10x+4|-x}{x^2-16}$

   c $\lim_{x\to 1} \frac{2x^2-5x+3}{\sqrt{11+5x}-3x-1}$

   d $\lim_{x\to -\infty} (\sqrt{36x^2-10x+3}+6x)$

2. **(10 points)** Use the formal definition of a derivative to calculate the derivative of $f(x)=\sqrt{x^3+x}$.

3. **(25 points)** Consider the sequence $\{a_n\}_{n=1}^{\infty}$ with

$$a_n = \frac{2n^2-2n+3}{n^2+n+1}$$

   a Is the sequence increasing or decreasing?

   b Calculate an upper and a lower bound for $\{a_n\}_{n=1}^{\infty}$.

   c Calculate $\lim_{n\to\infty} a_n$

   d Use the formal definition of a limit of a sequence to show that your solution to exercise c. is correct.

$\frac{n(n+1)}{2}$

$\sum 1+1+\frac{15}{13}$

$\sum_{i=1}^{n} i^n = \frac{n(n+1)(2n+1)}{6}$

$\sum_{i=2}^{n} r^{i-1} = \frac{r^n-1}{r-1}$

$i \neq 1$

4. **(20 points)**

a Calculate the 2[nd] degree Taylor polynomial P_2 for the function

$$f(x) = x\sqrt{x^2+1}$$

in $x = 0$. Then use this to approximate $\sqrt{2}$.

b Determine an interval for the error of the approximation in exercise a.

5. **(15 points)** Calculate

a $\int (x^2-4)\sqrt[5]{3x+5}\,dx$

b $\int \frac{2x^3+13x^2-33x-30}{x^3+3x^2-10x}\,dx$

c $\int \frac{\sqrt{4x^2-36}}{x^4}\,dx$

6. **(10 points)** Give the general solution to the following differential equation:

$$y''(t) - 2y'(t) + 5y(t) = 15t^2 - 22t - 10$$

2 solutions

$y = Ae^{r_1 t} + Ae^{r_2 t}$

1 solution

$y = Ae^{rt} + Ate^{rt}$

$ar^2 + br + c = 0$

$y = Ae^{rt}\cos(\omega t) + A$

DIFFERENTIATION RULES

$$\frac{d}{dx}\Big(f(x)+g(x)\Big) = f'(x)+g'(x)$$

$$\frac{d}{dx}\left(\frac{1}{f(x)}\right) = -\frac{f'(x)}{\big(f(x)\big)^2}$$

$$\frac{d}{dx}\Big(cf(x)\Big) = cf'(x)$$

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x)-f(x)g'(x)}{\big(g(x)\big)^2}$$

$$\frac{d}{dx}\Big(f(x)g(x)\Big) = f'(x)g(x)+f(x)g'(x)$$

$$\frac{d}{dx}f\Big(g(x)\Big) = f'\Big(g(x)\Big)g'(x)$$

ELEMENTARY DERIVATIVES

$$\frac{d}{dx}\frac{1}{x} = -\frac{1}{x^2}$$

$$\frac{d}{dx}e^x = e^x$$

$$\frac{d}{dx}\sin x = \cos x$$

$$\frac{d}{dx}\sec x = \sec x\tan x$$

$$\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$$

$$\frac{d}{dx}a^x = a^x\ln a \quad (a>0)$$

$$\frac{d}{dx}\cos x = -\sin x$$

$$\frac{d}{dx}\csc x = -\csc x\cot x$$

$$\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$$

$$\frac{d}{dx}x^r = rx^{r-1}$$

$$\frac{d}{dx}\ln x = \frac{1}{x} \quad (x>0)$$

$$\frac{d}{dx}\tan x = \sec^2 x$$

$$\frac{d}{dx}\cot x = -\csc^2 x$$

$$\frac{d}{dx}|x| = \operatorname{sgn} x = \frac{x}{|x|}$$

TRIGONOMETRIC IDENTITIES

$$\sin^2 x+\cos^2 x = 1$$

$$\sec^2 x = 1+\tan^2 x$$

$$\csc^2 x = 1+\cot^2 x$$

$$\sin(x\pm y) = \sin x\cos y \pm \cos x\sin y$$

$$\sin 2x = 2\sin x\cos x$$

$$\cos 2x = 2\cos^2 x - 1 = 1-2\sin^2 x$$

$$\sin(-x) = -\sin x$$

$$\sin(\pi - x) = \sin x$$

$$\sin\left(\frac{\pi}{2}-x\right) = \cos x$$

$$\cos(x\pm y) = \cos x\cos y \mp \sin x\sin y$$

$$\sin^2 x = \frac{1-\cos 2x}{2}$$

$$\cos(-x) = \cos x$$

$$\cos(\pi - x) = -\cos x$$

$$\cos\left(\frac{\pi}{2}-x\right) = \sin x$$

$$\tan(x\pm y) = \frac{\tan x\pm\tan y}{1\mp\tan x\tan y}$$

$$\cos^2 x = \frac{1+\cos 2x}{2}$$

QUADRATIC FORMULA

Integrals of tangent, cotangent, secant, and cosecant

$$\int \tan x\,dx = \ln|\sec x| + C,$$

$$\int \cot x\,dx = \ln|\sin x| + C = -\ln|\csc x| + C,$$

$$\int \sec x\,dx = \ln|\sec x + \tan x| + C,$$

$$\int \csc x\,dx = -\ln|\csc x + \cot x| + C = \ln|\csc x - \cot x| + C.$$