Calculus 1

January 29, 2009, 14:00-17:00

Clearly indicate how you came to your solution/conclusion.

Good luck with the exercises!

1. **(20 points)** Calculate

   a $\lim_{x \to 7} \dfrac{-x^2 + 9x - 14}{2x^2 - 10x - 30}$

   b $\lim_{x \to 1} \dfrac{|5x^3 - 3x^2 + x - 4| - |x|}{x^2 - 3x + 2}$

   c $\lim_{x \to -2} \dfrac{\sqrt{11 + 5x} - x - 3}{3x + 6}$

   d $\lim_{x \to \infty} (\sqrt{16x^2 - 12x + 3} - 4x)$

2. **(10 points)** Use the formal definition of a derivative to calculate the derivative of $f(x) = \sqrt[3]{2x + 1}$.

3. **(25 points)** Consider the sequence $\{a_n\}_{n=1}^{\infty}$ with

$$a_n = \frac{n^2 - 3n + 1}{2n^2 + n}.$$

   a Is the sequence increasing and/or decreasing?

   b Calculate an upper and a lower bound for $\{a_n\}_{n=1}^{\infty}$.

   c Calculate $\lim_{n \to \infty} a_n$.

   d Use the formal definition of a limit of a sequence to show that your solution to exercise c. is correct.

4. **(10 points)** Calculate the second degree Taylor polynomial P_2 for the function

$$f(x) = \sin(\sqrt{\pi x})$$

   in $x = \pi$.

5. **(10 points)** Show that the equation

$$4x^2 + 4x = 4 + \sqrt{3x + 6}$$

   has at least 2 solutions on $[-2, 1]$.

6. **(15 points)** Calculate

   a $\int x\sqrt{x^2 - 1}\, dx$

   b $\int \dfrac{x^2 + 2x + 5}{x^2 + 4x - 21}\, dx$

   c $\int \dfrac{1}{x\sqrt{4x^2 + 16}}\, dx$

7. **(10 points)** Give the general solution to the following differential equation:

$$y''(t) + 3y'(t) - 4y(t) = 8t^2 - 40t + 21$$

DIFFERENTIATION RULES

$$\frac{d}{dx}\Big(f(x)+g(x)\Big) = f'(x)+g'(x)$$

$$\frac{d}{dx}\Big(cf(x)\Big) = cf'(x)$$

$$\frac{d}{dx}\Big(f(x)g(x)\Big) = f'(x)g(x)+f(x)g'(x)$$

$$\frac{d}{dx}\left(\frac{1}{f(x)}\right) = -\frac{f'(x)}{\Big(f(x)\Big)^2}$$

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x)-f(x)g'(x)}{\Big(g(x)\Big)^2}$$

$$\frac{d}{dx}f\Big(g(x)\Big) = f'\Big(g(x)\Big)g'(x)$$

ELEMENTARY DERIVATIVES

$$\frac{d}{dx}\frac{1}{x} = -\frac{1}{x^2}$$

$$\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$$

$$\frac{d}{dx}x^r = rx^{r-1}$$

$$\frac{d}{dx}e^x = e^x$$

$$\frac{d}{dx}a^x = a^x \ln a \quad (a>0)$$

$$\frac{d}{dx}\ln x = \frac{1}{x} \quad (x>0)$$

$$\frac{d}{dx}\sin x = \cos x$$

$$\frac{d}{dx}\cos x = -\sin x$$

$$\frac{d}{dx}\tan x = \sec^2 x$$

$$\frac{d}{dx}\sec x = \sec x\tan x$$

$$\frac{d}{dx}\csc x = -\csc x\cot x$$

$$\frac{d}{dx}\cot x = -\csc^2 x$$

$$\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$$

$$\frac{d}{dx}|x| = \operatorname{sgn}x = \frac{x}{|x|}$$

TRIGONOMETRIC IDENTITIES

$$\sin^2 x+\cos^2 x = 1$$

$$\sin(-x) = -\sin x$$

$$\cos(-x) = \cos x$$

$$\sec^2 x = 1+\tan^2 x$$

$$\sin(\pi-x) = \sin x$$

$$\cos(\pi-x) = -\cos x$$

$$\csc^2 x = 1+\cot^2 x$$

$$\sin\left(\frac{\pi}{2}-x\right) = \cos x$$

$$\cos\left(\frac{\pi}{2}-x\right) = \sin x$$

$$\sin(x\pm y) = \sin x\cos y \pm \cos x\sin y$$

$$\cos(x\pm y) = \cos x\cos y \mp \sin x\sin y$$

$$\tan(x\pm y) = \frac{\tan x\pm\tan y}{1\mp\tan x\tan y}$$

$$\sin 2x = 2\sin x\cos x$$

$$\sin^2 x = \frac{1-\cos 2x}{2}$$

$$\cos^2 x = \frac{1+\cos 2x}{2}$$

$$\cos 2x = 2\cos^2 x - 1 = 1-2\sin^2 x$$

QUADRATIC FORMULA

Integrals of tangent, cotangent, secant, and cosecant

$$\int \tan x\,dx = \ln|\sec x|+C,$$

$$\int \cot x\,dx = \ln|\sin x|+C = -\ln|\csc x|+C,$$

$$\int \sec x\,dx = \ln|\sec x+\tan x|+C,$$

$$\int \csc x\,dx = -\ln|\csc x+\cot x|+C = \ln|\csc x-\cot x|+C.$$