CA 2, Testexam 2009

1. $\frac{\partial g}{\partial u} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} = ye^{xy}\cdot 3\sin v + xe^{xy}\cdot 4v^2 =$

$= 12v^2u\cdot \exp(12u^2v^2\sin v)\cdot \sin v +$

$+ 3u\sin v \exp(12u^2v^2\sin v)\cdot 4v^2 = \ldots$

$\frac{\partial g}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v} = ye^{xy}\cdot 3u\cos v + xe^{xy}\cdot 8vu =$

$= (12v^2u^2\cos v + 24u^2v\sin v)\exp(12u^2v^2\sin v).$

2. $\frac{\partial f}{\partial x} = e^{x^2-y^2} + 2x^2e^{x^2-y^2} = (1+2x^2)e^{x^2-y^2}$

$\frac{\partial f}{\partial y} = -2xye^{x^2-y^2}.$

$f_1(1,1) = 3, \quad f_2(1,1) = -2.$

Tangent plane:

$$\boxed{z-1 = 3(x-1) - 2(y-1)}$$

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3\. a) To find the intersection point, solve

$$\begin{cases} s^3 - s = t^3 - t \\ s^4 - 5s^2 = t^4 - 5t^2 \end{cases} \to \begin{cases} s^3 - s - t^3 + t = 0 \\ s^4 - 5s^2 - t^4 + 5t^2 = 0 \end{cases}$$

$$\begin{cases} (s-t)(s^2 + t^2 + ts - 1) = 0 \\ (s-t)(s+t)(s^2+t^2-5) = 0 \end{cases}$$

We only want solutions with $s \neq t$:

$$\begin{cases} s^2 + t^2 + ts - 1 = 0 & (1) \\ (s+t)(s^2+t^2-5) = 0 & (2) \end{cases}$$

From (2): $s = -t$ (I) or $s^2 + t^2 = 5$ (II)

Case (I): Substitute $s = -t$ in (1): $t^2 = 1$, hence $t = 1$ or $t = -1$. Yields $(s,t) = (1,-1)$ or $(s,t) = (-1,1)$

Case (II): $s^2 + t^2 = 5$: substitute in (1): $st = -4$, hence $s^2t^2 = 16$. Substitute $s^2 = 5 - t^2$:

$$(5-t^2)t^2 = 16 \to t^4 - 5t^2 + 16 = 0 : \text{ no solution}$$

It follows that the only intersection point is at $t = -1$ and $t = 1$, which corresponds to $(0,0)$.

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b) Vertical tangent line: necessary $\frac{dx}{dt} = 0$:
$3t^2 - 1 = 0$, $t = \pm \frac{1}{3}\sqrt{3}$

We have $\frac{dy}{dt} = 4t^3 - 10t = 2t(2t^2 - 5)$, hence $\frac{dy}{dt} \neq 0$ for $t = \pm\sqrt{3}$. It follows that indeed the tangent line is vertical for $t = \pm \frac{1}{3}\sqrt{3}$.

Horizontal tangent line: necessary: $\frac{dy}{dt} = 0$
$2t(2t^2 - 5) = 0 \rightarrow t = 0 \vee t = \pm\sqrt{5/2} = \pm\frac{1}{2}\sqrt{10}$
Notice that $\frac{dx}{dt} \neq 0$ for these 3 values of t, hence the tangent line is indeed horizontal for these values of t.

Conclusion: vertical at points $(0,4)$, $\left(\frac{3}{4}\sqrt{10}, -\frac{1}{4}\right)$, $\left(-\frac{3}{4}\sqrt{10}, -\frac{1}{4}\right)$
horizontal at points $\left(-\frac{2}{9}\sqrt{3}, \frac{22}{9}\right)$, $\left(\frac{2}{9}\sqrt{3}, \frac{22}{9}\right)$

c) $A = \left|\int_{-1}^{1} x(t)\, y'(t)\, dt\right| = \left|\int_{-1}^{1} (t^3 - t)(4t^3 - 10t)\, dt\right|$

$= \int_{-1}^{1} 2t^2(t^2 - 1)(2t^2 - 5)\, dt =$

$= \int_{-1}^{1} 2t^2(2t^4 - 5t^2 - 2t^2 + 5)\, dt = 2\int_{-1}^{1} 2t^6 - 7t^4 + 5t^2\, dt =$

$= 4\int_{0}^{1} 2t^6 - 7t^4 + 5t^2\, dt = 4\left(\frac{2}{7}t^7 - \frac{7}{5}t^5 + \frac{5}{3}t^3\right)\Big|_0^1$

$= 4\left(\frac{2}{7} - \frac{7}{5} + \frac{5}{3}\right) = 4\left(\frac{30}{105} - \frac{147}{105} + \frac{175}{105}\right) =$

$= 4 \cdot \frac{58}{105} = \frac{232}{105}$

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4. 

I $\int_0^4 \left(\int_{y=0}^{y=x} 4e^x - 5\sin y \, dy \right) dx =$

$\int_0^4 \left[4ye^x + 5\cos y \right]_{y=0}^{y=x} dx =$

$\int_0^4 4xe^x + 5\cos x - 5 \, dx =$

$= 4(x-1)e^x + 5\sin x - 5x \Big|_0^4 = 12e^4 + 5\sin 4 - 20 + 4$

$= 12e^4 + 5\sin 4 - 16$

II $\int_{y=0}^4 \left(\int_{x=y}^4 4e^x - 5\sin y \, dx \right) dy =$

$\int_0^4 \left[4e^x - 5x\sin y \right]_{x=y}^{x=4} dy =$

$\int_0^4 4e^4 - 20\sin y - 4e^y + 5y\sin y \, dy =$

$4e^4 \cdot y + 20\cos y - 4e^y + 5(\sin y - y\cos y) \Big|_0^4 =$

$16e^4 + 20\cos 4 - 4e^4 + 5\sin 4 - 20\cos 4 - 20 + 4 =$

$12e^4 + 5\sin 4 - 16$

5. Skip.

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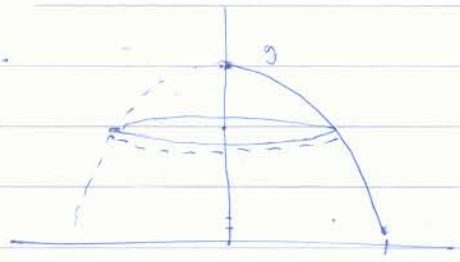
6.



b.1 $\int_0^3 \pi x^2 |f'(x)| \,dx =$

$2\pi \int_0^3 x^3 \,dx = \frac{1}{2}\pi x^4 \Big|_0^3 = \frac{81\pi}{2}.$

b2. $\int_0^9 \pi \left(\sqrt{9-y}\right)^2 dy = \pi \int_0^9 9-y \,dy = \pi\left(9y - \frac{1}{2}y^2\right)\Big|_0^9 =$

$= \pi\left(81 - \frac{81}{2}\right) = \frac{81\pi}{2}.$

a

$\int_0^3 2\pi x\,(9-x^2)\,dx =$

$2\pi \int_0^3 9x - x^3 \,dx =$

$2\pi \left(\frac{9}{2}x^2 - \frac{1}{4}x^4\right)\Big|_0^3 =$

$2\pi\left(\frac{81}{2} - \frac{81}{4}\right) = \frac{81\pi}{2}$

7.a) $z = y^2$. $z' = 2yy'$, $z'' = 2(y')^2 + 2yy''$

Then : $z' + z'' = 0$

b) $\lambda^2 + \lambda = 0 \Rightarrow \lambda = 0, -1.$ General solution for z:

$z = C_1 + C_2 e^{-t}$ $\quad C_1 \geq 0,\ C_2 \geq -C_1$

$y = \pm\sqrt{C_1 + C_2 e^{-t}}$