GENERAL

Differentiation Rules:

$\frac{d}{dx}\left(f(x)+g(x)\right)=f'(x)+g'(x)$	Sum rule
$\frac{d}{dx}\left(f(x)g(x)\right)=f'(x)g(x)+f(x)g'(x)$	Product rule
$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right)=\frac{f'(x)g(x)-f(x)g'(x)}{\left(g(x)\right)^2}$	Quotient rule –follows from Product rule
$\frac{d}{dx}\left(cf(x)\right)=cf'(x)$	Follows from Product rule
$\frac{d}{dx}\left(\frac{1}{f(x)}\right)=-\frac{f'(x)}{\left(f(x)\right)^2}$	Follows from Quotient rule
$\frac{d}{dx}f\left(g(x)\right)=f'\left(g(x)\right)g'(x)$	Chain rule –first differentiate the outside formula, then inside formula

Elementary Derivatives:

$\frac{d}{dx}\,x^r=rx^{r-1}$	The generic power function derivative, <u>based on variable x</u>
$\frac{d}{dx}\sqrt{x}=\frac{1}{2\sqrt{x}}$	Follows from above generic derivative
$\frac{d}{dx}\frac{1}{x}=-\frac{1}{x^2}$	Follows from above generic derivative
$\frac{d}{dx}\,a^x=a^x\ln a \quad (a>0)$	The generic power function derivative, <u>based on constant a</u>
$\frac{d}{dx}e^x=e^x$	Follows from above generic derivative (note that ln e=1)
$\frac{d}{dx}\ln x=\frac{1}{x} \quad (x>0)$	
$\frac{d}{dx}\sin x=\cos x$	
$\frac{d}{dx}\cos x=-\sin x$	The "co-" functions have explicit minus signs in their derivatives
$\frac{d}{dx}\tan x=\sec^2 x$	
$\frac{d}{dx}\sec x=\sec x\,\tan x$	
$\frac{d}{dx}\csc x=-\csc x\,\cot x$	The "co-" functions have explicit minus signs in their derivatives
$\frac{d}{dx}\cot x=-\csc^2 x$	The "co-" functions have explicit minus signs in their derivatives
$\frac{d}{dx}\sec^{-1} x=\frac{1}{\sqrt{1-x^2}}$	
$\frac{d}{dx}\tan^{-1} x=\frac{1}{1+x^2}$	
$\frac{d}{dx}\lvert x\rvert=sgn\,x=\frac{x}{\lvert x\rvert}$	Signum function, note the "absolute" signs. sgn (x)=1 if x>0 sgn (x)=-1 if x<0 sgn (x)=undefined if x=0

tan x = sin x / cos x

sec x = 1 / cos x

csc x = 1 / sin x

cot x = 1 / tan x

See note below

See note below

Trigonometric Identities:

$\sin^2 x + \cos^2 x = 1$	$\sec^2 x = 1 + \tan^2 x$	
	$\csc^2 x = 1 + \cot^2 x$	
$\sin(-x) = -\sin x$		Watch the signs!
$\cos(-x) = \cos x$		Watch the signs!
$\sin(\pi - x) = \sin x$	Remember: sin 0 = 0, $\sin\frac{\pi}{2} = 1$ *and* $\sin \pi = 0$	
$\cos(\pi - x) = -\cos x$	Remember: cos 0 = 1, $\cos\frac{\pi}{2} = 0$ *and* $\cos \pi = 0$	
$\sin\left(\frac{\pi}{2} - x\right) = \cos x$		
$\cos\left(\frac{\pi}{2} - x\right) = \sin x$		
$\sin((x \pm y)) = \sin x \cos y$ $\pm \cos x \sin y$	$\sin 2x = \sin(x + x)$ $= 2 \sin x \cos x$ Watch the signs!	$\sin^2 x = \frac{1 - \cos 2x}{2}$ Remember: $\sin^2 x + \cos^2 x = 1$
$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$	$\cos 2x = 2\cos^2 x - 1$ $= 1 - 2\sin^2 x$ Watch the signs!	$\cos^2 x = \frac{1 + \cos 2x}{2}$ Remember: $\sin^2 x + \cos^2 x = 1$
$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$	Watch the signs!	
$\sin^2 x = \frac{1}{2}(1 - cos2x)$	$\cos^2 x = \frac{1}{2}(1 + cos2x)$	Double-angle formulas, see page 307

Quadratic Formula:

If $Ax^2 + Bx + C = 0,\ then:$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Properties of the definite integral:

$\int_a^a f(x)dx = 0$	**Interval is zero (a -> a), so the integral is zero as well.**
$\int_b^a f(x)dx = -\int_a^b f(x)dx = 0$	**Reversing limits of integration changes the sign of the integral**
$\int_a^b (Af(x) + Bg(x))dx = A\int_a^b f(x)dx + B\int_a^b g(x)dx$	**Integral depends linearly on integrand, thus sum can be split in individual components (and constants A and B can be brought outside the integral).**
$\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)\,dx$	**For the same function f(x), intervals can be merged into a single interval a -> c.**
$\int_a^b f(x)dx \le \int_a^b g(x)dx$	**If a ≤ b and f(x) ≤ g(x) for the interval a ≤ x ≤ b**
$\left\lvert\int_a^b f(x)dx\right\rvert \le \int_a^b \lvert f(x)dx\rvert$	**Only if a ≤ b (triangle inequality)**
$\int_{-a}^a f(x)dx = 0$	**Only applies to an <u>odd</u> function i.e.** $(f(-x) = -f(x))$
$\int_{-a}^a f(x)dx = 2\int_0^a f(x)dx$	**Only applies to an <u>even</u> function i.e.** $(f(-x) = f(x))$

Integration rules:

$\int f'(g(x))g'(x)dx = f(g(x)) + C$	**Method of substitution which is the <u>inverse</u> of the Chain Rule. Do not forget the generic constant C.**
$\int U(x)dV(x) = U(x)V(x) - \int V(x)dU(x)$	**Integration by parts, which is the <u>inverse</u> of the Product Rule for differentiation**
$\int_a^b f'(x)dx = f(b) - f(a)$	
$\frac{d}{dx}\int_a^x f(t)dt = f(x)$	**Fundamental Theorem of Calculus (see page 297).** **The integral is a function of x because x is the (variable) upper limit. If you then differentiate the integral, you simply get the function as a result. Note that x <u>must</u> be the upper limit , otherwise the interval limits must be reversed which changes the sign of the integral.**
$\frac{d}{dx}\int_a^{g(x)} f(t)dt = f(g(x))\,g'(x)$	**Builds the chain rule into the Fundamental theorem (upper interval limit is now function of x)**
$\frac{d}{dx}\int_{h(x)}^{g(x)} f(t)dt = f(g(x))\,g'(x) - f(h(x))h'(x)$	**Expands the chain rule into the Fundamental theorem to include lower interval limit (function of x as well)**

Some elementary integrals:

$\int 1\,dx = x + C$	$\int x\,dx = \frac{1}{2}x^2 + C$
$\int x^2\,dx = \frac{1}{3}x^3 + C$	$\int \frac{1}{x^2}dx = -\frac{1}{x} + C$
$\int \sqrt{x}\,dx = \frac{2}{3}\,x^{\frac{3}{2}} + C$	$\int \frac{1}{\sqrt{x}}\,dx = 2\sqrt{x} + C$
$\int x^r\,dx = \frac{1}{r+1}\,x^{r+1} + C$	
$\int \frac{1}{x}dx = ln\vert x\vert + C$	
$\int e^x\,dx = e^x + C$	$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$
$\int a^x\,dx = \frac{a^{\;x}}{ln\,a} + C$	
$\int \sin x\,dx = -\cos x + C$	$\int \sin ax\,dx = -\frac{1}{a}\cos ax + C$
$\int \cos x\,dx = \sin x + C$	$\int \cos ax\,dx = \frac{1}{a}\sin ax + C$
$\int sec^2\,x\,dx = tan\,x + C$	$\int sec^2\,ax\,dx = \frac{1}{a}\,tan\,ax + C$
$\int csc^2\,x\,dx = -\,cot\,x + C$	$\int csc^2\,ax\,dx = -\frac{1}{a}\,cot\,ax + C$
$\int \sec x\,tan\,x\,dx = \sec x + C$	$\int \sec ax\,tan\,ax\,dx = \frac{1}{a}\sec x + C$
$\int \csc x\,cot\,x\,dx = -\csc x + C$	$\int \csc ax\,cot\,ax\,dx = -\frac{1}{a}\csc ax + C$
$\int tan\,x\,dx = ln\,\vert sec\,x\vert + C$	
$\int cot\,x\,dx = ln\,\vert sin\,x\vert + C$	
$\int sec\,x\,dx = ln\,\vert sec\,x + tan\,x\vert + C$	
$\int csc\,x\,dx = ln\,\vert csc\,x - cot\,x\vert + C$	
$\int \frac{1}{\sqrt{a^2 - x^2}}dx = sin^{-1}\frac{x}{a} + C\;\;(a > 0)$	$\int \frac{1}{a^2 + x^2}dx = \frac{1}{a}tan^{-1}\frac{x}{a} + C$

Some logarithmic properties:

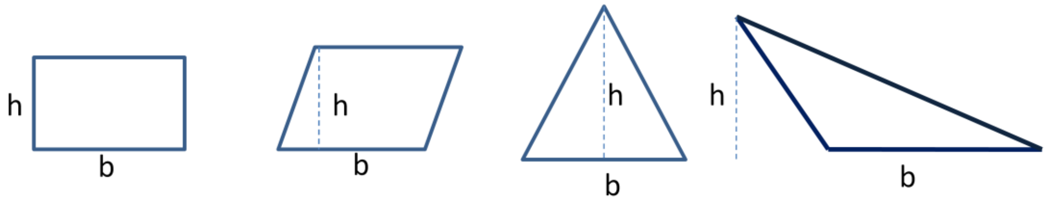
$e^y = x$ (where e is the base) is equivalent to y =$log_e\,x$ = ln x

ln ab = ln a + ln b

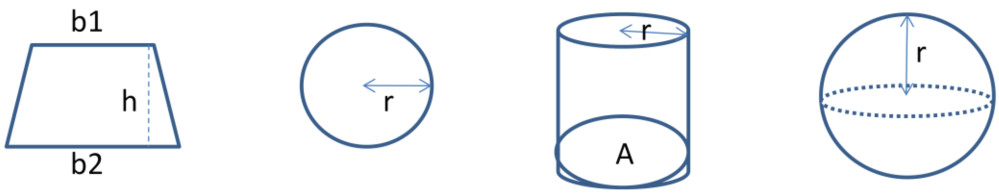
ln $\frac{a}{b} = \ln a - \ln b$

Geometric formulas:

A=area, b=base, h=height
C=circumference, V=volume, S= surface area



$A = bh$ $A = bh$ $A = \frac{1}{2} bh$ $A = \frac{1}{2} bh$



$A = \frac{1}{2}(b1+b2)\,h$

$A = \pi r^2$
$C = 2\pi r$

$V = Ah = \pi r^2 h$
$S = Ch = 2\pi rh$

$V = 4/3\,\pi r^3$
$S = 4\,\pi r^2$

You get from A to C (m^2 to m) or V to S (m^3 to m^2) by differentiation !

Inverse Tangent

Definition 1: Inverse Tangent

$\mathbf{y = tan^{-1}x}$ if and only if $\mathbf{tan\ y = x}$ where $\mathbf{-\pi/2}$ is less than $\mathbf{y}$ is less than $\mathbf{\pi/2}$.

Lemma: $\mathbf{sec^2x = 1 + tan^2\ x}$

Proof:

(1) By definition $\mathbf{sec\ x = 1/cos\ x}$

(2) So, $\mathbf{sec^2x = 1/(cos^2\ x)}$

(3) $\mathbf{sin^2x + cos^2x = 1}$ [See Corollary 2, here]

(4) $\mathbf{1/(cos^2\ x) = (cos^2x + sin^2x)/(cos^2) =}$

$\mathbf{= cos^2x/(cos^2x) + (sin^2x)/(cos^2) = 1 + tan^2x.}$

QED

Theorem: $\mathbf{D\ tan^{-1}\ x = 1/(1 + x^2)}$

Proof:

(1) Let $\mathbf{y = tan^{-1}\ x}$

(2) Then $\mathbf{tan\ y = x}$ [See Definition 1 above]

(3) $\mathbf{d(tan\ y)/dx = d(x)/dx = 1}$

(4) $\mathbf{d(tan\ y)/dx = (sec^2y)dy/dx = 1}$ [See Theorem 1, here]

(5) $\mathbf{dy/dx = 1/sec^2y}$

(6) Using Lemma 1 above, we have:

$dy/dx = 1/(1 + \tan^2 y)$

(6) Using step #2, this gives us:

$dy/dx = 1/(1 + \tan^2 y) = 1/(1 + x^2)$ [Since **$x = \tan y$**]

QED

References

- Edwards & Penny, Calculus and Analytic Geometry

1.2 LIMITS OF FUNCTIONS

Informal definition of a limit: $\lim_{x\to a} f(x) = L$

Function f approaches the limit L as x approaches a

Tricky examples are those where the denominator approaches zero (no apparent feasible solution)!!!
This requires re-writing of the problem e.g.

$$\lim_{x\to 4} \frac{(\sqrt{x}-2)}{x^2-16}$$

can be re-written to $\lim_{x\to 4} \frac{1}{(x+4)(\sqrt{x}+2)}$ in which case it can be solved (see page 64)

Informal definition of left and right limits: $\lim_{x\to a-} f(x) = L$ **(left limit)**

$\lim_{x\to a+} f(x) = L$ **(right limit)**

Limit rules:

If $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$ and k is a constant, then:

$\lim_{x\to a}[f(x) + g(x)] = L + M$	Limit of a sum
$\lim_{x\to a}[f(x) - g(x)] = L - M$	Limit of a difference
$\lim_{x\to a} f(x)g(x) = LM$	Limit of a product
$\lim_{x\to a} kf(x) = kL$	Limit of a multiple
$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{L}{M}$	Limit of a quotient
$\lim_{x\to a}[f(x)]^{\frac{m}{n}} = L^{\frac{m}{n}}$	Limit of a power

Limits of polynomials and rational functions:

If P(x) is a polynomial and a is any real number, then:
$\lim_{x\to a} P(x) = a$

If P(x) and Q(x) are polynomials and $Q(a) \neq 0$, then:
$\lim_{x\to a} \frac{P(x)}{Q(x)} = \frac{P(a)}{Q(a)}$

Squeeze theorem:

Assume that $f(x) \le g(x) \le h(x)$ holds for all x in some open interval containing a (except possibly x=a itself). Suppose also that $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$, then:

$$\lim_{x\to a} g(x) = a$$

1.3 LIMITS AT INFINITY and INFINITE LIMITS

Limits at infinity and negative infinity (informal definition)

If the function f is defined on an interval (0,∞) and if we can ensure that f(x) is as close as we want to the number L by taking x large enough, then we say that f(x) approaches the limit L as x approaches infinity, *and we write*

$$\lim_{x\to\infty} f(x) = L$$

If the function f is defined on an interval (-∞, 0) and if we can ensure that f(x) is as close as we want to the number M by taking x large enough in absolute value, then we say that f(x) approaches the limit M as x approaches negative infinity, *and we write*

$$\lim_{x\to-\infty} f(x) = M$$

Tricky examples are those where the numerator and denominator approach infinity (no apparent feasible solution)!!!
This requires re-writing of the problem e.g.

$\lim_{x\to+/-\infty} \frac{(5x+2)}{(2x^3-1)}$ can be re-written to $\lim_{x\to+/-\infty} \frac{(5/x^2+2/x^3)}{\left(2-\left(\frac{1}{x}\right)\right)}$ in which case it can be solved (see page 64)

4.6 FINDING ROOTS OF EQUATIONS

Linear equations: ax + b = 0 → x = - $\frac{b}{a}$

Quadratic equations: ax^2 + bx + c = 0 → $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

In addition to the bisection method (section 1.4), there are 2 other approximation methods:

Newton's method formula: $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

Fixed point iteration: $x_{n+1} = f(x_n)$ for n = 0,1,2,3,4...... (watch convergence!)
You have to write the function as f(x) = x and then solve.
Example: Find the root of the equation cos x = 5x. Write the function as f(x) = $\frac{1}{5}$ cosx =x and solve.

4.7 LINEAR APPROXIMATIONS

Linearization of the function f about a is the function L defined by:
L(x) = f(a) + f'(a) (x-a). This provides an approximation for values of f near a.

The error (deviation from true value) is: E(x) = $\frac{f''(s)}{2}(x-a)^2$ (i.e. second derivative for linear equation)

4.8 TAYLOR POLYNOMIALS

Approximation becomes more accurate when the polynomial is extended:

$$\boldsymbol{P_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots\ldots + \frac{f^n(a)}{n!}(x-a)^n}$$

Error is: $E_n(x) = \frac{f^{(n+1)}(s)}{(n+1)!}(x-a)^{(n+1)}$

5.1 SUMS AND SIGMA NOTATION

Sigma notation: $\sum_{i=m}^{n} f(i) = f(m) + f(m+1) + f(m+2) + \cdots \ldots \ldots \ldots F(n)$

Practice change of index. Example page 276:
$\sum_{j=3}^{17} \sqrt{1+j^2} = \sum_{i=1}^{15} \sqrt{1+(i+2)^2}$

Summation formulas for evaluating sums through closed form expressions:

$$\sum_{i=1}^{n} 1 = 1 + 1 + 1 + \cdots \ldots \ldots + 1 = n$$
$$\sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots \ldots .. + n = \frac{n(n+1)}{2}$$
$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + 3^2 + \cdots \ldots .. + n^2 = \frac{n(n+1)(2n+1)}{6}$$
$$\sum_{i=1}^{n} r^{i-1} = 1 + r + r^2 + \cdots \ldots .. + r^{n-1} = \frac{r^n - 1}{(r-1)} \text{ if } r \neq 1$$

5.2 AREAS AS LIMITS OF SUM

You need the above summation formulas to help calculating areas, which you do by summing up all the small rectangle areas underneath the curve, see example 2 on page 282.

In example 4 on page 284 you have to re-write the limit expression and find the y=f(x)=1-x expression yourself. Practice on a few examples.

You can always cross-check the answers if you know the integral of the function.

5.3 THE DEFINITE INTEGRAL

Riemann sums: Upper and lower, depending how you define the starting point for the Y function (above or below the curve, see example 1 on page 286).

Definite integral: I = $\int_a^b f(x)dx$

Where:

- **∫** is the integral sign (resembles S which is limit of sum)
- **a** and **b** are the limits of integration (lower/upper limits resp)
- function **f** is the integrand
- dx is the differential of x (replaces Δx in Riemann sums) differential tells you the variable of integration.
- Replacing the variable of integration does not change the value of the integral, thus $\int_a^b f(x)dx = \int_a^b f(t)dt$

5.3 PROPERTIES OF THE DEFINITE INTEGRAL

value of the