Universiteit Maastricht
Faculty of General Sciences
Knowledge Engineering

Exam

Logic

Wednesday 31 May 2006

This exam is NOT an open-book exam.
You will get an auxiliary sheet with the most important formulas.

Clarify all your answers sufficiently!
Indicate in semantic tableaux exactly which reduction rules you are using.
In the case of derivations, indicate exactly from which formula('s) each formula is derived and if the conditions are met (if applicable).

This exam contains 9 assignments. For each assignment will be indicated the maximum amount of points that you can obtain for a right answer. The exam mark is the number of points for all assignments together, divided by 10.

1. (10 points)
   About Eric we know the following:
   i. If he does not use his bicycle, then he takes the bus or the train.
   ii. If he takes the train, then he does not take the bus.
   iii. If he takes the bus, then he does not use his bicycle.
   iv. If he does not take the train, then he uses his bicycle.

   Does Eric take the bus?
   Determine the answer to this question (yes or no) with the aid of a truth table (model elimination).

2. Semantic tableaux propositional logic. (12 points)
   a. Explain the $\neg_R$ reduction rule.
   b. Given is the consequence $p \vee r, \neg((q \to p) \to r)$ / $\neg(q \to r)$. Investigate the validity of this consequence with the aid of a semantic tableau. If this consequence is not valid, give all the counterexamples.
   c. Investigate by means of a semantic tableau whether the set of formulas $\{p \vee q, p \to r, q \to s, \neg(r \vee s)\}$ is inconsistent (if the set is consistent, then give a valuation that is a model of the set).

3. Natural deduction (propositional logic) (12 points).
   Prove by means of natural deduction:
   a. $p \to r, q \to s \vdash (q \wedge p) \to (r \wedge s)$
   b. $p \vee q, p \to r \vdash (q \to r) \to r$
   c. $p \to r, q \to \neg r \vdash p \to \neg q$

4. (10 points)
   Translate the following formula into an equivalent formula in prenex form:
   $((\exists x\ P(x,y)) \vee \neg(\forall x\ ((\forall z\ Q(x,z)) \to (\exists z\ \forall y\ R(y,z)))))$

5. (12 points)
Consider the domain of natural numbers and sets of natural numbers, with the following predicates:
Ox means that x is an odd number, Exy means that x is an element of y, and Rxy means that $x < y$. The predicate = may also be used.
Now write each of the following sentences as a formula of predicate logic:

    a. There is an odd number.
    b. The set S exists of precisely three odd numbers.
    c. There exist arbitrarely large odd numbers.

6. Semantic tableaux predicate logic. (12 points)
    a. Explain the $\forall_L$ reduction rule.
    Investigate the validity of the following consequences by means of a semantic tableau. If a consequence is not valid, give at least one counterexample.
    b. $\forall x\,(Ax \vee Bx), \neg\forall x\, Ax \,/\, \neg\forall x\forall y\,(Bx \rightarrow Ay)$
    c. $\forall x\exists y(Kxy \rightarrow Bx) \,/\, \forall x((\forall y Kxy) \rightarrow Bx)$

7. (12 points)
    a. Explain the $\forall$I rule. Why is the condition essential?
    Prove with the aid of natural deduction:
    b. $\forall x\,(Px \rightarrow Rx), \forall x\,(Qx \rightarrow Rx), \forall x\forall y\,(Px \vee Qy) \vdash \forall x\, Rx$
    c. $\exists x\,(Ax \wedge Bx), \neg\exists x\,(Bx \wedge Cx) \vdash \neg\forall x\,(Ax \rightarrow \mathrm{C}x)$

8. (10 points)
A formula φ of the proposition logic is called *negation free* if φ does not contain any negation symbol.
For each formula φ of the proposition logic it holds that:
either there exists a negation-free formula ψ such that $\varphi \equiv \psi$,
or there exists a negation-free formula χ such that $\varphi \equiv \neg\chi$
(stated otherwise, either φ or $\neg\varphi$ is logically equivalent to a negation-free formula).
    a. Show this for the formulas $\neg p \rightarrow q$, $p \rightarrow \neg q$, $\neg p \rightarrow \neg q$, $\neg p \vee q$, $\neg p \vee \neg q$, $p \wedge \neg q$ and $r \rightarrow (\neg\neg p \wedge \neg q)$.
    b. Describe a general method to transform φ into ψ or $\neg\chi$, or give a proof with formula induction.
    c. Does this property also hold for the predicate logic? Motivate your answer.

9. (10 points)
Consider the language with only the binary predicate symbol R (thus, no function symbols, no constants). Let M_1 and M_2 be two models, such that for both models the corresponding structure has the domain $\mathbb{R}$ (the real numbers). Let the interpretation functions of M_1 and M_2 be such that
$V_{M1,b}(R(x,y)) = (b(x) < b(y))$, and
$V_{M2,b}(R(x,y)) = ((b(x))^2 < (b(y))^2)$
for all look-up tables b.

Find a formula φ such that M_1 is a model of φ and M_2 is a not a model of φ. Show that your formula indeed satisfies the requirements.