Maastricht University
Faculty of Humanities and Sciences
Knowledge Engineering

Logic 2007/2008

Resit Exam

Wednesday, July 2, 2008, 18.00–21.00h

This exam is NOT an open-book exam. An auxiliary sheet with the most important formulas is added to the exam.

Clarify all your answers sufficiently! Indicate in semantic tableaux exactly which reduction rules you are using. In the case of derivations, indicate exactly from which formula(s) each formula is derived and if the conditions are met (if applicable).

This exam contains nine assignments. For each assignment the maximum score that you can obtain for a correct answer is indicated. The final score for the exam is the sum of the scores for the individual assignments, divided by 10.

1. (10 points)
   Thomas goes to Las Vegas and plays roulette, blackjack, and poker.

   i. If he does not win at roulette, he wins at blackjack or poker.
   ii. He does not win at both roulette and blackjack.
   iii. If he does not win at blackjack, he does not win at poker.
   iv. If he wins at roulette, he wins at poker.

   Does Thomas win at roulette? Does he win at blackjack?
   Determine the answers with the aid of a truth table (model elimination).

2. (12 points)
   Determine the answers to the following questions using semantic tableaux:

   (a) Consider the consequence $p \to (q \wedge r), \neg p \to \neg(q \vee r)\ /\ q \to \neg r$.
   Is this consequence valid? If it is not valid, then give all counterexamples.
   (b) Consider the formula $(((p \to q) \to r) \wedge (p \to r)) \to r$.
   Is this formula a tautology? If it is not a tautology, then give all counterexamples.

3. (12 points)
   Prove by means of natural deduction:

   (a) $p \to (q \wedge r), q \to s \vdash p \to (r \wedge s)$
   (b) $p \to q, \neg q \vee r \vdash p \to r$
   (c) $(p \wedge q) \to \neg r, (p \vee r) \to q, q \to (r \to p) \vdash \neg r$

4. (10 points)
   Transform the following formula to an equivalent formula in prenex form:
   $\neg \forall z(\neg Pxz \wedge \exists x((\forall y Qyz \to \forall x \exists y Rxyz) \to Sxy))$

5. (12 points)
   Consider the domain of natural numbers $\mathbb{N}$. Px means that x is prime. Gxy means that x is greater than y. Dxy means that x is divisible by y. a is the constant 1. The predicate $=$ may also be used. Now write each of the following sentences as a formula of the predicate logic:

   (a) There are at least two prime numbers.

   (b) There is no highest prime number.

   (c) A prime number is divisible by precisely two numbers, namely 1 and itself.

6. (12 points)
   Investigate the validity of the following consequences by means of semantic tableaux. If a consequence is not valid, give at least one counterexample.

   (a) $\exists x Ax \to \forall x Bx \ / \ \forall x \forall y (Ax \to By)$

   (b) $\exists x (Ax \to Bx) \ / \ \exists x (Ax \leftrightarrow Bx)$

7. (12 points)
   Prove by means of natural deduction:

   (a) $\forall x (Ax \to \forall x Bx) \vdash \forall x \forall y (Ax \to By)$

   (b) $\forall x (Ax \vee \neg \exists y Bxy), \exists x (\neg Ax \to Bxx) \vdash \exists x Ax$

8. (10 points)
   Given is a model $M = (\mathbf{D}, I)$.

   Structure $\mathbf{D} = \langle \{\spadesuit, \heartsuit, \clubsuit, \diamondsuit\}, \{(\spadesuit, \heartsuit), (\spadesuit, \clubsuit), (\spadesuit, \diamondsuit), (\heartsuit, \clubsuit), (\diamondsuit, \clubsuit)\}, \varnothing \rangle$.

   Under interpretation function I, Rxy is true if relation $(I(x), I(y))$ exists in $\mathbf{D}$.

   (a) Is M a model of $\forall x \exists y Rxy$? Explain.

   (b) Is M a model of $\exists x \forall y Rxy$? Explain.

   (c) Is M a model of $\exists x \forall y \exists z (Rxy \to Ryz)$? Explain.

   (d) For which look-up tables b does it hold that $M, b \models \exists x \exists z (Rxy \wedge Rxz \wedge Ryz)$?

9. (10 points)
   Consider the propositional logic. Let formula ψ be a subformula of formula φ if ψ occurs in the construction tree of φ.

   For example, if $\varphi = ((\neg p \vee q) \to p)$, then φ has five subformulas, namely p, q, $\neg p$, $(\neg p \vee q)$, and $((\neg p \vee q) \to p)$.

   Using formula induction, prove that the total number of subformulas of any formula φ is less than or equal to the sum of the number of proposition letters and the number of logical operators that occur in φ.

Good luck!