Topic 1: Error Analysis and Algebraic Equations

Graded homework questions: 4(b), 6(c). Answer each question on a *separate* sheet of paper!

Exercises which you should do by hand (with a calculator, but not a computer) are marked **H**. Exercises in which the use of a computer is central are marked **C**. Optional extra exercises are marked **E**. Advanced optional exercises are marked **A**.

Round-Off Error and Computer Arithmetic

1. Compute the absolute error and relative error in approximations of p by p^*.

a. $p = \pi$, $p^* = 3.14$

b. $p = \pi$, $p^* = 3.1416$

c. $p = e^{10}$, $p^* = 22000$

d. $p = 10^{\pi}$, $p^* = 1400$

2. Use three-digit (significant figures) rounding arithmetic to perform the following calculations. Compute the absolute error and relative error with the exact value determinated to at least five digits.

a. $133 + 0.921$

b. $133 - 0.499$

c. $(121 - 0.327) - 119$

d. $(121 - 119) - 0.327$

e. $\big((13 \div 14) - (6 \div 7)\big) \div (2 \times e - 5.4)$

f. $\big(\pi - (22 \div 7)\big) \div (1 \div 17)$

Hint: In Matlab, use `rnd(x,n)` (from the file `rnd.m` available on EleUM) to round x to n significant figures. You can define `r=@(x)rnd(x,4)` so that `r(x)` rounds x to 4 significant figures.

Errors in Scientific Computing

3. (i) Use four-digit rounding arithmetic to find the roots of the following quadratic equations. Use both forms of the quadratic formula. (ii) Compute the absolute and relative errors for these approximations.

a. $\frac{1}{3}x^2 - \frac{123}{4}x + \frac{1}{6} = 0$

b. $1.002x^2 + 11.01x + 0.01265 = 0$

4. Evaluate each of the following polynomial at the given point x using decimal arithmetic rounded to 4 significant figures using (i) direct evaluation and (ii) nested (Horner) form. (iii) Compare your answers to the exact value.

a. $p(x) = 1.013x^5 - 5.262x^3 - 0.01732x^2 + 0.8389x - 1.912, \quad x = 2.289.$

b. $p(x) = x^4 - 5.4x^3 + 10.56x^2 - 8.954x + 2.795, \quad x = 1.3.$

Solutions of Equations of One Variable

C5. Use the built-in Matlab commands `fsolve`, `fzero` and `roots` to find all solutions of the polynomial equation $x^3 + 4x^2 - 9 = 0$. What is the difference between `fsolve` and `fzero`? For what functions can `roots` be used?

Note: Consult the Matlab Help system for information on how to use these commands.

H6. Use (i) the bisection method, (ii) the secant method and (iii) Newton's method to find solutions accurate to within (i) 10^{-1} and (ii,iii) 10^{-3} for the following problems:

a. $x^2 - 3 = 0$ on $[1, 2]$.

b. $x^3 - 2x^2 - 5 = 0$ on $[1, 4]$.

c. $5x - 4 = \sin x$ on $[0, \pi/2]$.

d. $\sqrt{x} - \cos x = 0$ on $[0, 1]$.

Note: Make sure you evaluate cosine in *radians*!

7. Apply the bisection method with a tolerance 10^{-1} to attempt to solve the following equation, starting with each of the given intervals:

$$x^3 - 7x^2 + 14x - 6 = 0 \quad \text{on} \quad [0, 13],\ [0, 4],\ [1, 4],\ [1, 3],\ [3, 4].$$

What happens in each case? Give all roots of the equations.

8. What happens if you apply the bisection method to find the root(s) of $\tan(x) - x$ in $[1, 2]$? *Hint:* Plot the functions if you can't see what is going on here!

C9. The `bisection_root.m` file (on EleUM) defines a function

`c=bisection_root(f,a,b,e)`

which computes a root of $f(x) = 0$ in $[a, b]$ using the bisection method, with an error guaranteed to be less than e. At each step, it outputs $[a, c, b,\ f(c), \epsilon = (b - a)/2]$.

Use the `bisection_root` function to estimate the solutions of the following equations to an accuracy of 10^{-6}.

a. $2x \cos 2x - (x - 2)^2 = 0$ on $[2, 3]$ and on $[3, 4]$.
b. $(x - 2)^2 - \ln x = 0$ on $[1, 2]$ and on $[3, 4]$.
c. $e^x - 3x^2 = 0$ on $[0, 1]$ and on $[3, 5]$.
d. $\sin x - e^{-x} = 0$ on $[0, 1]$, on $[3, 4]$ and on $[6, 7]$.

What happens if you apply `bisection_root` to $[2, 4]$ in (a)?

C10. Write a Matlab function which uses the secant method to estimate a solution of the equation $f(x) = 0$:

`r=secant(f,p0,p1,eps,nmax)`

where f is the function, p_0, p_1 are the initial estimates, ϵ is the desired error bound and $n_{\max}$ is a maximum number of steps. The function should return an estimate r of the root.

Write a similar function which uses Newton's method:

`[r,err]=newton(f,df,p0,nmax,e,d)`

where *df* is the derivative of f, p_0 is the initial estimate, e is the desired error bound and d a bound on $f(r)$. The function should return $[r, \epsilon]$, where r is the estimate of the root, and ϵ an estimate of the error.

Test your functions on the problems of Question 9 using a tolerance of 10^{-6}.

C11. Use the secant method and/or Newton's method to find all solutions of the following equations accurate to within 10^{-5}.

a. $x^4 - 7x^2 + 3x + 1 = 0$.
b. $4x \cos(2x) - (x - 2)^2 = 0$ in $[0, 8]$

12. Use Newton's method to find the roots of the function $f(x) = x^3 - 6x^2 + 7x + 2$ starting at $p_0 = 1$. What happens? Draw the graph of the function and illustrate this graphically.

Find all roots of the function. Investigate how different starting points yield different roots.

13. The function $f(x) = \tan(3x) - 5$ has a unique zero in the interval $[0, 0.5]$. Use 10 iterations of (i) the bisection method, (ii) the secant method and (iii) Newton's method starting at both 0 and 0.5, to approximate this root. Which method is most successful, and why?