Topic 3: Numerical Integration and Differentiation

Graded homework questions: 1(b), {13,15}(b).

Integration

1. Use (i) the (composite) midpoint rule, (ii) the trapezoidal rule, and (iii) Simpson's rule to approximate the integrals

a. $\displaystyle\int_{0.5}^{1} x^4\,dx$ **b.** $\displaystyle\int_{1}^{1.6} \frac{2x}{4-x^2}\,dx$

Use $n=1$, $n=2$ and $n=4$ for the midpoint and trapezoidal rules, and $n=2$ and $n=4$ for Simpson's rule. Compare your results with the exact values (a) $31/160$ and (b) $\ln(25/12)$.

How do your results correspond to what you would expect given the order of the methods?

2. Given the function f at the following values:

x	1.8	2.0	2.2	2.4	2.6
$f(x)$	3.12014	4.42569	6.04241	8.03014	10.4668

Approximate $\int_{1.8}^{2.6} f(x)\,dx$ as accurately as possible using (i) the midpoint rule, (ii) the trapezoidal rule and (iii) Simpson's rule.

3. Use (i) the midpoint rule, (ii) the trapezoidal rule and (iii) Simpson's rule with the indicated values of n, or the indicated h to approximate the following integrals.

a. $\displaystyle\int_{1}^{2} x\ln x\,dx, \quad n=4$ **b.** $\displaystyle\int_{-2}^{2} x^3e^x\,dx, \quad n=4$

c. $\displaystyle\int_{0}^{2} \frac{2}{x^2+4}\,dx, \quad n=6$ **d.** $\displaystyle\int_{0}^{\pi} x^2\cos x\,dx, \quad n=6$

e. $\displaystyle\int_{0}^{2} e^{-x^2}\,dx, \quad h=0.25$ **f.** $\displaystyle\int_{0}^{2} x^2e^{-x^2}\,dx, \quad h=0.25$

4. Compute the Romburg integral $R_{4,4}$ approximating

a. $\displaystyle\int_{0}^{5} e^{-x^2}\,dx$ **b.** $\displaystyle\int_{0}^{4} \frac{\sin x}{x}\,dx$

Compare your answers with those obtained by using Simpson's rule with $n=2,4,8$, and explain any correspondences between your results.

Note: Take $\sin(0)/0 = \lim_{h\to 0}\sin(h)/h = 1$.

A5. For the following integrals, compute the Simpson's rule approximation $S(a,b)$, $S(a,(a+b)/2)$ and $S((a+b)/2,b)$, and verify the estimate given in the adaptive quadrature error estimate result.

a. $\displaystyle\int_{0}^{1} x^2e^{-x}\,dx.$ **b.** $\displaystyle\int_{0}^{\pi} x\sin(x)\,dx.$

Use adaptive quadrature to find approximations to within 10^{-3} for the integrals.

A6. Use the error formula to find a bound for the error for each integral in Question 1, and compare the bound to the actual error.

A7. For each of the functions below, find upper bounds for

$$\sup_{\xi\in[a,b]}|f''(\xi)| \quad\text{and}\quad \sup_{\xi\in[a,b]}|f^{(4)}(\xi)|.$$

Hence the values of n and h required to approximate the following integrals using (i) the trapezoidal and (ii) Simpson's rules to within 10^{-4}, and evaluate the integral to this accuracy.

a. $\int_0^\pi \sin x\,dx$ **b.** $\int_0^2 e^{2x}\sin 3x\,dx$

C8. Use Matlab's built-in `integral` command to evaluate the following integrals:

a. $\displaystyle\int_0^5 e^{-x^2/2}\,dx$ **b.** $\displaystyle\int_0^{4\pi} \frac{\sin x}{x}\,dx$

C9. Write Matlab functions to approximate $\int_a^b f(x)\,dx$ using (i) the midpoint rule, (ii) the trapezoidal rule and (iii) Simpson's rule with a given value of n. Test your code by computing the following integrals with the given values of n.

a. $\displaystyle\int_0^2 e^{2x}\sin 3x\,dx, \quad n = 8, 16$ **b.** $\displaystyle\int_1^3 \frac{x}{x^2+4}\,dx, \quad n = 8, 16$

c. $\displaystyle\int_3^5 \frac{1}{\sqrt{x^2-4}}\,dx, \quad n = 8, 16$ **d.** $\displaystyle\int_0^{3\pi/8} \tan 2x\,dx, \quad n = 8, 16$

C10. Write a Matlab function to compute the Romburg approximation $R_{n,n}$ to $\int_a^b f(x)\,dx$. Test your answer by computing $R_{n,n}$ for various values of n for the integrals given in Question 9. Compare and comment on the number of function evaluations needed for the various methods to obtain a given accuracy.

C11. Write a Matlab function to estimate $\int_a^b f(x)\,dx$ with accuracy ϵ using an adaptive version of the composite Simpson's rule. Test your code by estimating the integrals below to accuracy 10^{-6}.

c. $\displaystyle\int_0^\pi x\cos(x^2)\,dx$ **d.** $\displaystyle\int_{1/3\pi}^{4/pi} \sin(1/x)\,dx.$

Also use Simpson's rule with $n = 4, 6, 8, \ldots$, until successive approximation to the following integrals agree to within 10^{-6}. Determine the number of nodes required. Use adaptive quadrature to approximate the integral to within 10^{-6} and count the number of nodes. Did adaptive quadrature produce any reduction in the number of function evaluations required?

Differentition

12. Use the (two-point) forward-difference formulae and backward-difference formulae to determine each missing entry in the following tables.

a.

x	$f(x)$	$f'(x)$
0.5	0.4794	
0.6	0.5646	
0.7	0.6442	
0.8	0.7174	

b.

x	$f(x)$	$f'(x)$
0.0	0.00000	
0.2	0.74140	
0.4	1.3718	

13. Use the most accurate three-point formula to determine each missing entry in the following tables.

a.

x	$f(x)$	$f'(x)$
1.1	9.025013	
1.2	11.02318	
1.3	13.46374	
1.4	16.44465	

b.

x	$f(x)$	$f'(x)$
8.1	16.94410	
8.3	17.56492	
8.5	18.19056	
8.7	18.82091	

c.

x	$f(x)$	$f'(x)$
2.9	-4.827866	
3.0	-4.240058	
3.1	-3.496909	
3.2	-2.596792	

d.

x	$f(x)$	$f'(x)$
2.0	3.6887982	
2.1	3.6905701	
2.2	3.6688192	
2.3	3.6245909	

14. Use the five-point differentiation formulae to determine, as accurately as possible, approximations for each missing entry in the following tables.

a.

x	$f(x)$	$f'(x)$
2.1	−1.709847	
2.2	−1.373823	
2.3	−1.119214	
2.4	−0.9160143	
2.5	−0.7470223	
2.6	−0.6015966	

b.

x	$f(x)$	$f'(x)$
−3.0	9.367879	
−2.8	8.233241	
−2.6	7.180350	
−2.4	6.209329	
−2.2	5.320305	
−2.0	4.513417	

15. Use the three-point formula to determine the second derivatives for the data given in Question 13.

C16. Estimate the first derivative of $f(x) = 1/(2 - x^2)$ at $x = 1$ using (i) the three-point centred-difference formula, the (ii) three-point forward difference formula, and (iii) the five-point centred difference formula, and the second derivative at $x = 1$ using (iv) the three-point fomulae. Use $h = 10^{-1}$, $h = 10^{-2}$, $h = 10^{-3}$ and $h = 10^{-6}$. Compute the errors in the approximations and verify the order of each method.

17. Let ϵ denote a bound on the total error made in evaluating a function f, and m denotes a bound for the third derivative of f. Show that the total error in estimating $f'(x)$ using the three-point centred-difference formula is

$$e(h) = \frac{\epsilon}{h} + \frac{h^2}{6}m.$$

Which value of h should you choose to minimise the error?

A18. The data for Question 13(a) was taken from the function $f(x) = e^{2x}$ and for 13(b) from $f(x) = x \ln x$. Re-compute the derivatives using the exact values of the function and determine the effect of rounding to 5 decimal places in the table. Compute the actual errors and compare with the error bounds given by the error formulae.

19. Use the following data to approximate $f'(3)$ and $f''(3)$ as accurately as possible.

x	1.0	2.0	3.0	4.0	5.0
$f(x)$	2.4142	2.6733	2.8974	3.0976	3.2804

Find bounds for the errors, given that $\sup_{\xi\in[1,5]} |f^{k)}(\xi)| \leq M_k$ with $M_1 = 0.3$, $M_2 = 0.06$, $M_3 = 0.04$, $M_4 = 0.04$, $M_5 = 0.05$, $M_6 = 0.09$, respectively.

Note: The data was generated by $f(x) = 1 + \sqrt{\tfrac{2}{5}(3 + 2x)} = \sqrt{0.8x + 1.2} + 1.0$.

A20. Show that if f is 5-times continuously differentiable, then

$$f'(x) = \frac{-3f(x-h) - 10f(x) + 18f(x+h) - 6f(x+2h) + f(x+3h)}{12h} - \frac{h^4}{20} f^{(5)}(\xi)$$

for some ξ between $x - h$ and $x + 3h$.

Show that if f is 6-times continuously differentiable, then

$$f''(x) = \frac{-f(x-2h) + 16f(x-h) - 30f(x) + 16f(x+h) - f(x+2h)}{12h^2} - \frac{h^4}{90} f^{(6)}(\xi)$$

for some ξ between $x - 2h$ and $x + 2h$.

Use these formula to find better estimates to $f'(x)$ for the data entries next to the endpoints in Question 14, and to estimate the second derivatives.

Note: Other five-point second-derivative formulae are

$$f''(x) = \frac{11f(x-h) - 20f(x) + 6f(x+h) + 4f(x+2h) - f(x+3h)}{12h^2} - \frac{19}{360}\, h^4 f^{(6)}(\xi)$$

$$f''(x) = \frac{35f(x) - 104f(x+h) + 114f(x+2h) - 56f(x+3h) + 11f(x+4h)}{12h^2} - \frac{119}{90}\, h^4 f^{(6)}(\xi)$$

Applications

21. In a multivariable calculus and in statistics courses, it is shown that

$$\int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-x^2/2\sigma^2}\,dx = 1$$

for any positive σ. The function

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-x^2/2\sigma^2}$$

is the *normal density function* with *mean* $\mu = 0$ and *standard deviation* σ. The probability that a randomly chosen value described by this distribution lies in $[a, b]$ is given by $\int_a^b f(x)dx$. Approximate to within 10^{-5} the probability that a randomly chosen value described by this distribution will lie in

a. $[-2\sigma, 2\sigma]$

b. $[-3\sigma, 3\sigma]$

22. A car laps a race track in 84 seconds. The speed of the car at each 6-second interval is determined using a radar gun and is given from the beginning of the lap, in feet per second, by the entries ion the following table.

Time	0	6	12	18	24	30	36	42	48	54	60	66	72	78	84
124	134	148	156	147	133	121	109	99	85	78	89	104	116	123	

How long is the track?

23. To determine the thermal characteristics of disk brakes, you need the "area averaged lining temperature", T, of the brake pad from the equation

$$T = \frac{\int_{r_e}^{r_0} T(r) r \theta_p \, dr}{\int_{r_e}^{r_0} r \theta_p \, dr}$$

where r_e represents the radius at which the pad-disk contact begins, r_0 represents the outside radius of the pad-disk contact, and θ_p represents the angle subtended by the sector brake pads. The temperature $T(r)$ at each point of the pad is obtained numerically from analyzing the head equation. Suppose $r_e = 0.308$ft, $r_0 = 0.478$ft, $\theta_p = 0.7051$ radians, and the temperatures gien in the following table have been calculated at the various points on the disk. Approximate T.

r (ft)	$T(r)$ (°F)	r (ft)	$T(r)$ (°F)	r (ft)	$T(r)$ (°F)
0.308	640	0.376	1034	0.444	1204
0.325	794	0.393	1064	0.461	1222
0.342	885	0.410	1114	0.478	1239
0.359	943	0.427	1152		

24. The differential equation

$$m\ddot{u}(t) + ku(t) = F_0 \cos \omega t$$

describes a spring-mass system with mass m, spring constant k, and no applied damping. The term $F_0 \cos \omega t$ describes a periodic external force applied to the system. The solution to the equation when the system is initially at rest ($\dot{u}(0) = u(0) = 0$) is

$$u(t) = \frac{2F_0}{m(\omega_0^2 - \omega^2)} \sin \frac{(\omega_0 - \omega)}{2} t \sin \frac{(\omega_0 + \omega)}{2} t, \quad \text{where} \quad \omega_0 = \sqrt{\frac{k}{m}} \neq \omega.$$

Sketch the graph of u when $m = 1$, $k = 9$, $F_0 = 1$, $\omega = 2$, and $t \in [0, 2\pi]$. Approximate $\int_0^{2\pi} u(t)\,dt$ to within 10^{-4}.

25. The study of light diffraction at a rectangular aperture involves the Fresnel integrals

$$c(t) = \int_0^t \cos \tfrac{\pi}{w} w^2 \, dw \quad \text{and} \quad s(t) = \int_0^t \sin \tfrac{\pi}{2} w^2 \, dw\,.$$

Construct a table of values for $c(t)$ and $s(t)$ that is accurate to within 10^{-4} for values of $t = 0.1, 0.2, \ldots, 1.0$.