Topic 4: Ordinary Differential Equations

Graded homework questions: 4(b), 17(a).

1. Use (i) Euler's method and (ii) the second-order Taylor method to approximate the solutions for each of the following initial-value problems, and compute the actual error at each step.

a. $\dot{y} = e^{-t} - y/2$, for $0 \leq t \leq 2$, $y(0) = 0$ with $h = 1.0$ and $h = 0.5$; solution $y(t) = 2(e^{-t/2} - e^{-t})$.

b. $\dot{y} = 1 + (t - y)^2$, for $2 \leq t \leq 3$, $y(2) = 1$ with $h = 0.5$ and $h = 0.25$; solution $y(t) = t + 1/(1 - t)$.

2. Use (i) the midpoint method, (ii) the trapezoid (modified Euler) method, (iii) Heun's second-order method, (iv) Heun's third-order method, and (v) the fourth-order Runge-Kutta method to solve each of the problems in Question 1. Compute the actual error after the first step and after the final step, and compare the accuracy of the different methods.

3. Use (i) the two-stage and (ii) three-stage Adams-Bashforth methods to approximate the solutions to the initial-value problems in Question 1. Use exact starting values. Compare the results to the actual values.

4. Use the Adams-Bashforth two-stage method as a predictor and the Adams-Moulton two-stage method as a corrector to approximate the solutions to the inital-value problems in Question 1. Use an appropriate Runge-Kutta method to determine the starting values.

C5. Consider the differential equation

$$\frac{dy}{dt} = e^{-t} - y^2; \quad y(0) = 0\,. \tag{$\dagger$}$$

Use Matlab's builtin commands `ode23` and `ode45` to compute an approximation to the solution over the time interval $[0, 10]$.

Adapt your code to use fixed step-sizes $h = 0.2$, $h = 0.1$ and $h = 0.05$. Compare your answers with the exact solution, which has $y(1) = 0.503346658225$, $y(2) = 0.478421766451$, $y(5) = 0.237813428537$ and $y(10) = 0.110790590981$. What is the order of each method?

C6. A second-order differential equation $\ddot{x} = f(t, x, \dot{x})$ can be converted into a system of first-order differential equations

$$\dot{x} = v; \qquad \dot{v} = f(t, x, v).$$

The resulting state vector $\mathbf{y} = (x; v)$ satisfies

$$\dot{\mathbf{y}} = \begin{pmatrix} \dot{x} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} v \\ f(t, x, v) \end{pmatrix} = \begin{pmatrix} y_2 \\ f(t, y_1, y_2) \end{pmatrix} =: \mathbf{f}(t, \mathbf{y}).$$

Use Matlab's `ode45` method to solve the *Airy equation* $\ddot{x}(t) = tx(t)$ with $x(0) = 1$ and $\dot{x}(0) = 0$. What happens to the solution as $t \to \infty$?

C7. Write Matlab functions implementing (i) Euler's method, (ii) Heun's second-order method (iii) Heun's third-order method, and (iv) the fourth-order Runge-Kutta method. At each step, display t_i, w_i, $f(t_i, w_i)$ and any $k_{i,j}$.

Use the methods you have developed to approximate the solutions for each of the following initial-value problems, and compare the results to the actual values.

a. $\dot{y} = 1 + (y/t) + (y/t)^2$, for $1 \leq t \leq 3$, $y(1) = 0$ with $h = 0.1$; solution $y(t) = t \tan(\ln t)$.

b. $\dot{y} = 1 - y^2$, for $0 \leq t \leq 2$, $y(0) = 0$ with $h = 0.2$; solution $y(t) = 2/(1 + e^{-2t}) - 1$.

c. $\dot{y} = -y^2/(t + 1)$, for $1 \leq t \leq 2$, $y(1) = 1/\ln 2$, with $h = 0.1$; solution $y(t) = 1/\ln(t + 1)$.

d. $\dot{y} = -ty + 4t/y$, for $0 \leq t \leq 1$, $y(0) = 1$, with $h = 0.1$; solution $y(t) = \sqrt{4 - 3e^{-t^2}}$.

C8. Write Matlab functions implementing (i) the explicit three-stage Adams-Bashforth method and (ii) the three-stage Adams-Bashforth method as a predictor and the three-stage Adams-Moulton method as a corrector to solve the initial value problems in Question 7. In each case, bootstrap using a suitable Runge-Kutta method.

AC9. Adapt your predictor-corrector method so to use the previously-computed value $f(t_i, \tilde{w}_i)$ to predict $\tilde{w}_{i+1}$, instead of computing $f(t_i, w_i)$. Make sure you only evaluate f *once* per time step. Determine experimentally the order of the method.

A10. Derive 2-stage Adams-Bashforth formulae for doubling and halving the step size.

11. Use the three-stage Adams-Moulton method to approximate the solutions to the initial-value problems in Question 1. In each case, use exact starting values and solve the implicit algebraic equation for w_{i+1}. Compare the results to the actual values.

T12. Write the solution of the differential equation $\dot{y} = e^{-t}\cos(2t)$ as an integral. Solve the equation for $t \in [0, 3]$ with $y(0) = 0$ using the fourth-order Runge-Kutta method with step-size $h = 0.5$. Which integration method is this equivalent to?

T13. How many function evaluations are needed to solve the initial value problem (†) using a fourth-order Runge-Kutta method with a step size of 0.01, compared with a four-stage Adams-Bashforth method, or the three-stage Adams-Moulton method? Which method provides the best accuracy for the *same* number of function evaluations?

14. For each of the following initial-value problems, compute two steps of the adaptive Bogacki-Shampine method with given tolerance. Compare the approximate solution to the actual solution.

a. $\dot{y} = e^t\sqrt{2 - y^2}$ with $y(0) = 0$ and $TOL = 10^{-3}$; exact solution $y(t) = \sqrt{2}\sin(e^t - 1)$.

b. $\dot{y} = \dot{y} = -y + 1 - y/t$ with $y(1) = 1$ and $TOL = 10^{-2}$; exact solution $y(t) = 1 + (e^{1-t} - 1)/t$.

C15. Given the initial-value problem

$$\dot{y} = 2y/t + t^2 e^t, \quad 1 \leq t \leq 2, \quad y(1) = 0$$

with the exact solution $y(t) = t^2(e^t - e)$:

a. Use Euler's method with $h = 0.1$ to approximate the solution and compare it with the actual values of y.

b. Use the answers generated in (a) and linear interpolation to approximate the following values of y and compare them to the actual values.

i. $y(1.04)$ **ii.** $y(1.55)$ **iii.** $y(1.97)$.

c. Use the fourth-order Runge-Kutta method with $h = 0.1$ to approximate the solution and compare it with the actual values of y.

d. Use the answers generated in (c) and cubic spline interpolation to approximate y at the same values as in (b), and compare them to the actual values of y.

The use of interpolation to estimate the solution at times between explicitly computed time points is a common technique when solving differential equations, as it does not require additional function evaluation.

C16. Use (i) the fourth-order Runge-Kutta method for systems and (ii) the Adams fourth-order predictor-corrector method to approximate the solutions of the following systems of first-order differential equations and compare the results to the actual solutions.

a. $\dot{y}_1 = y_1(3 - y_1 - y_2)$ with $y_1(0) = 1$; $\dot{y}_2 = y_2(y_1 - 1)$ with $y_2(0) = 1$; $h = 0.2$, for $0 \leq t \leq 4$.

b. $\dot{x} = \sigma(y - x)$; $\dot{y} = x(\rho - z) - y$; $\dot{z} = xy - \beta z$ for $\alpha = 10$, $\beta = 8/3$, $\rho = 28$; with $x(0) = 1$, $y(0) - = z(0) = 0$ with for $0 \leq t \leq 10$.

C17. Use (i) the fourth-order Runge-Kutta method for systems and (ii) the Adams fourth-order predictor-corrector method to approximate the solutions of the following higher-order differential equations.

a. $\ddot{y} + \dot{y} + \sin(y) = \cos(t)$, for $0 \leq t \leq 10$ with $y(0) = \dot{y}(0) = 0$ and $h = 0.1$.

b. $\ddot{y} + 0.2\dot{y} + y(y^2 - 1) = 0.3\cos(t)$

AC18. Write a Matlab function implementing the fully-implicit three-stage Adams-Moulton method, using the secant method to solve the resulting algebraic equation

$$g(w_{i+1}) := w_{i+1} - \Big(w_i + \frac{h}{24}\big(9f(t_{i+1}, w_{i+1}) + 19f(t_i, w_i) - 5f(t_{i-1}, w_{i-1}) + f(t_{i-2}, w_{i-2})\big)\Big) = 0.$$

Apply you method to solve the initial value problem (†).

AC19. Write your own adaptive solver based on the Bogacki-Shampine method. Test your routine on the differential equation

$$\dot{x} = 1 - tx; \qquad x(0) = 0.5 \tag{†}$$

over the interval $t \in [0, 2]$.

C20. Solve the following stiff initial-value problems using (i) Euler's method, (ii) the fourth-order Runge-Kutta method, (iii) the Adams fourth-order predictor-corrector method, and (iv) the trapezoidal method. Also use (v) the backward Euler method and (vi) the implicit trapezoidal method, incorporating the secant method (or Newton's method) to solve for w_{i+1}. Comment on the stability of the methods. Compare the results withn the actual solution

a. $\dot{y} = -20(y - t^2) + 2t$, for $0 \leq t \leq 1$, with $y(0) = \frac{1}{3}$ and $h = 0.1$; actual solution $y(t) = t^2 + \frac{1}{3}e^{-20t}$.

b. $\dot{y} = 50/y - 50y$, for $0 \leq t \leq 1$, with $y(0) = \sqrt{2}$ and $h = 0.1$; actual solution $y(t) = (1 + e^{-100t})^{1/2}$.

C21. The Matlab solvers `ode23s` and `ode15s` can be used to solve stiff systems. Try these solvers on the initial-value problems of Question 20, and compare your results with the built-in solvers `ode23` and `ode45` for non-stiff systems.

T22. For the differential equation

$$\dot{y} = -\lambda y,$$

find a formula for w_{i+1} in terms of w_i using (i) the backward Euler method and (ii) the implicit trapezoidal method, (iii) Euler's method, and (iv) the method obtained using Euler's method as a predictor and the implicit trapezoidal rule as a corrector with step size h. Determine the stability radius of each method.

Applications

23. The irreversible chemical reaction in thich two molecules of solid potassium dichromate ($K_2Cr_7O_7$), two molecules of water (H_2O), and three atoms of solid sulphur (S) combine to yield three molecules of the gas sulphur dioxide (SO_2), four molecules of solid potassium hydroxide (KOH), and two moledules of solid chromic oxide (Cr_2O_3), can be represented symbolically by the *stoichiometric equation*:

$$2K_2Cr_2O_7 + 2H_2O + 3S \longrightarrow 4KOH + 2Cr_2O_3 + 3SO_2.$$

If n_1 molecules of $K_2Cr_2O_7$, n_2 molecules of H_2O, and n_3 molecules of S are originally available, the following differential equation describes the amount $x(t)$ of KOH after time t:

$$\frac{dx}{dt} = k\left(n_1 - \frac{x}{2}\right)^2 \left(n_2 - \frac{x}{2}\right)^2 \left(n_3 - \frac{3x}{4}\right)^3$$

where k is the rate constant of the reaction. If $k = 6.22\times10^{-19}$, $n_1 = n_2 = 2\times10^3$, and $n_3 = 3\times10^3$, use the Runge-Kutta method of order 4 to determine how many units of potassium hydroxide will have been formed after 0.2 s.

24. An electrical circuit consists of a capacitor of constant capacitance $C = 1.1$ farads in series with a resistor of constant resistance $R_0 = 2.1$ ohms. A voltage $\mathcal{E}(t) = 110 \sin t$ is applied at time $t = 0$. When the resistor heats up, the resistance becomes a function of the current I,

$$R(t) = R_0 + kI, \qquad \text{where } k = 0.9,$$

and the differential equation for I becomes

$$\left(1 + \frac{2k}{R_0} I\right) \frac{dI}{dt} + \frac{1}{R_0 C} I = \frac{1}{R_0 C} \frac{d\mathcal{E}}{dt}.$$

Find the current I after 2 s, assuming $I(0) = 0$.

AC25. Given the initial-value problem

$$\dot{y} = 2y/t + t^2 e^t, \quad 1 \le t \le 2, \quad y(1) = 0$$

with the exact solution $y(t) = t^2(e^t - e)$:

a. Use Euler's method with $h = 0.1$ to approximate the solution and compare it with the actual values of y.

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