Topic 5: Approximation of Functions

Graded homework questions: 4(d), 13. Answer each question on a *separate* sheet of paper!

Polynomial Approximation

1. Compute the linear least squares polynomial for the following data, estimate the missing value, and compute the total square error and the root-mean-square error. Graph the data and the approximation.

a.

x_i	0.00	0.25	050	0.75	1.00	0.60
y_i	1.0000	1.2840	1.6487	2.1170	2.7183	

b.

x_i	1.0	1.1	1.3	1.5	1.9	2.1	2.0
y_i	1.84	1.96	2.21	2.45	2.94	3.18	

Note: The root-mean-square error in the discrete case is $\left(\frac{1}{m}\sum_{i=1}^{m}(p(x_i)-y_i)^2\right)^{1/2}$, and in the continuous case is $\left(\frac{1}{b-a}\int_a^b(p(x)-f(x))^2dx\right)^{1/2}$.

2. Find the least squares polynomials of degrees 2 and 3 for the data in Question 1, estimate the missing value, and compute the root-mean-square error in each case.

3. Find the linear and quadratic least squares polynomial approximations to $f(x)$ on the indicated interval if

a. $f(x)=x^2-2x+3$, $[0,1]$;

b. $f(x)=x^3$, $[-1,1]$;

c. $f(x)=1/x$, $[1,3]$;

d. $f(x)=e^x$, $[0,2]$;

e. $f(x)=\frac{1}{2}\cos x+\frac{1}{3}\sin 2x$, $[0,1]$;

f. $f(x)=\ln x$, $[1,3]$.

4. Compute the coefficients of Legendre basis polynomials for the least squares polynomial approximations of degrees 1, 2 and 3 on the interval $[-1,1]$ for the following functions:

a. $f(x)=x^3$

b. $f(x)=1/(x+2)$

c. $f(x)=e^x$

d. $f(x)=\ln(x+2)$

Compute the total square errors. How does the difference in the errors relate to the coefficients?

C5. Use Matlab's `polyfit` function to compute least-squares approximations of degrees 1, 2 and 3 to the following data:

x	0.5	1.0	2.0	2.5	3.0	3.5	4.0
y	1.61	2.13	2.56	2.47	2.26	2.05	1.96

For each case, compute the root-mean-square error of the approximation over the data points, and estimate the value of y when $x=1.5$.

C6. Write a Matlab function to compute the coefficients a_0, a_1 of the linear least-squares approximation to data (x_i, y_i), $i=1,\ldots,n$.

Use your function to compute the linear least-squares approximation to the data in Question 5.

C7. Plot the Legendre polynomials P_k for $k=0,\ldots,6$.

Compute the coefficients c_k of P_k in the best sixth-degree least-squares approximation to the functions given in Question 4, and plot the resulting polynomials p.

Hint: The functions P_k can be computed using the recurrence relation

$$P_k(x)=\big((2k-1)xP_{k-1}(x)-(k-1)P_{k-2}(x)\big)/k.$$

8. Use the Gram-Schmidt process to construct the *Laguerre polynomials* L_k, which are orthogonal on the interval $[0, \infty)$ with respect to the weight function $w(x) = e^{-x}$.

9. Use the Laguerre polynomials calculated to compute the least squares polynomials of degree 1, 2, and 3 on the interval $[0, \infty)$ with respect to the weight function $w(x) = e^{-x}$ for the following functions.

a. $f(x) = x^2$ **b.** $f(x) = e^{-x}$

c. $f(x) = x^3$ **d.** $f(x) = e^{-2x}$

Fourier Series

10. Find the continuous least squares trigonometric polynomial $S_n(x)$ on $[-\pi, \pi]$ for $n = 1, 2, 3$ and the following functions.

a. $f(x) = x^2$; **b.** $f(x) = x^2 \cos x$; **c.** $f(x) = e^x$.

Compute the total square errors. How does the difference in the errors relate to the coefficients?

A11. Determine the discrete least squares trigonometric polynomial $S_n(x)$ on the interval $[-\pi, \pi]$ for the functions given in Question 10, using $m = 4, 8$ and $n = 3$. Compare your answers with those found previously.

12. Find the discrete least squares trigonometric polynomials $S_n(x)$ on $[-\pi, \pi]$ for the following data:

a.

x_i	$-\pi$	$-2\pi/3$	$-\pi/3$	0	$\pi/3$	$2\pi/3$	π	$\pi/2$
y_i	0.54	0.37	0.69	1.87	2.69	1.44	0.54	

b.

x_i	$-\pi$	$-3\pi/4$	$-\pi/2$	$-\pi/4$	0	$\pi/4$	$\pi/2$	$3\pi/4$	π	$\pi/3$
y_i	0.49	0.81	0.98	0.96	0.75	0.42	0.08	0.14	0.49	

Use your approximations to estimate the missing value.

C13. Compute the coefficients of the trigonometric polynomial $s_3(x)$ of degree $n = 3$ approximating the function $f(x) = \sin x/(2 + \sin x)$. Plot the graph of the function f and the Fourier expansion.

Hint: An integral $\int_a^b f(x)\,dx$ can be computed using `integrate(f,a,b)`.

C14. Compute the coefficients of the trigonometric polynomials approximating the data (x_i, y_i) below:

a.

x_i	$-\pi$	$-3\pi/4$	$-\pi/2$	$-\pi/4$	0	$\pi/4$	$\pi/2$	$3\pi/4$
y_i	0.17647	0.33194	0.47059	0.88653	0.76471	0.60924	0.47059	0.05464

b.

x_i	$-\pi$	$-3\pi/4$	$-\pi/2$	$-\pi/4$	0	$\pi/4$	$\pi/2$	$3\pi/4$
y_i	0.00000	0.06678	0.21630	0.57983	1.00000	0.57983	0.21630	0.06678

Comment on your results.

C15. Compute the Fourier coefficients b_m up to $m = 15$ for the wavelet

$$f(x) = \exp\bigl(3\cos(x)\bigr)\sin(5x).$$

Comment on how the coefficients vary. Which is the largest coefficient?

C16. **a.** Determine the trigonometric interpolating polynomial $S_4(x)$ of degree 4 for $f(x) = x^2 \sin x$ on the interval $[-\pi, \pi]$.

b. Compute $\int_0^1 S_4(x)\,dx$, and compare your approximation to $\int_0^1 x^2 \sin x\,dx$.

A17. Find the coefficients a_n, b_n of the Fourier series of the *square wave*, which is the 2π-periodic function satisfying

$$f(x) = \begin{cases} 0, & \text{if } -\pi \leq x \leq 0, \\ 1, & \text{if } 0 < x < \pi. \end{cases}$$

Write down the general form of the least-squares approximation s_n.

Plot S_n for $n = 1, 2, 3, \ldots$. What happens to the least-squares error as $n \to \infty$? What happens to the uniform error?

A18. Compute the coefficients of the Fourier series of the *sawtooth wave*, which is the 2π-periodic function satisfying

$$f(x) = x \text{ for } x \in [-\pi, +\pi).$$

By computing $\int_0^\pi f(x)^2\,dx$ and $\int_0^\pi s(x)^2\,dx$, find a formula for $\sum_{k=1}^{\infty} 1/k^2$.

AC19. Write a Matlab function to compute the discrete Fourier transform using the fast Fourier transform method. Compute the trigonometric interpolating polynomial of degree 4 and 6 on $[-\pi, \pi]$ using $2m = 2^6$ and 2^8 data points for the following functions.

a. $f(x) = \pi(x - \pi)$

b. $f(x) = |x|$

c. $f(x) = \cos \pi x - 2 \sin \pi x$

d. $f(x) = x \cos x^2 + e^x \cos e^x$

Applications

20. Hooke's law states that when a force is applied to a spring constructed of uniform material, the length of the spring is a linear function on the force that is applied; $k(l - E) = F(l)$.

a. Suppose that $E = 5.3 \in$, and that measurements are made of the length l in inches for applied weights $F(l)$ in punds, as given in the following table. Find the least squares approximation for k.

$F(l)$	2	4	6
l	7.0	9.4	12.3

b. Additional measurements are made, giving the following additional daa. use these data to compute a new least squares approximation for k. Which of (a) or (b) best fits the total experimental data?

$F(l)$	3	5	8	10
l	8.3	11.3	14.4	15.9

21. The following table lists the college grade-point averages (GPA) of 20 mathematics and computer science majors, together with the scores that these students received on the mathematics portion of the ACT (American College Testing Program) test while in high school. Plot these data, and find the equation of the least squares line for this data. Do you think that the ACT scores are a reasonable predictor of college grade-point averages?

ACT	28	25	28	27	28	33	28	29	23	27
GPA	3.84	3.21	3.23	3.63	3.75	3.20	3.41	3.38	3.53	2.03

ACT	29	28	27	29	21	28	27	26	30	24
GPA	3.75	3.65	3.87	3.75	1.66	3.12	2.96	2.92	3.10	2.81

Non-examinable

22. *Non-examinable* Determine all Padé approximations for $f(x) = e^{2x}$ of degree 2. Compare the results at $x_i = 0.2i$, for $i = 1, 2, 3, 4, 5$, with the actual values of $f(x_i)$.

23. *Non-examinable* To accurately approximate $\sin x$ and $\cos x$ for inclusion in a mathematical library, we first restrict their domains. Given a real number x, divide by π to obtain the relation

$$|x| = M\pi + s, \qquad \text{where } M \text{ is an integer and } |s| \leq \frac{\pi}{2}.$$

a. Show that $\sin x = \operatorname{sgn}(x)\,(-1)^M\ \sin s$.

b. Construct a rational approximation to $\sin s$ using $n = m = 4$. Estimate the error when $0 \leq |s| \leq \frac{\pi}{2}$.

c. Design an implementation of $\sin x$ using (a) and (b).

d. Repeat (c) for $\cos x$ using the fact that $\cos x = \sin(x + \pi/2)$.