DKE Numerical Mathematics
Exam 30 May 2012

Part I

During this part of the exam, you may *only* use the formula sheet provided, and an electronic calculator. You may *not* use a textbook or your own notes.

1. [**6 points**] Use two iterations of Newton's method to estimate the positive root of $x^2 - 3 = 0$, starting your iteration with $x_0 = 2$.

2. [**10 points**] Compute the values $z_1 = s''(1)$ and $z_2 = s''(2)$ for the cubic spline $s(x)$ interpolating the data

x	0	1	2	3
y	2	5	4	8

   using the natural spline boundary conditions $s''(0) = s''(3) = 0$. Write down the polynomial $s_1(x)$ giving the values of s for $1 \leq x \leq 2$.

3. [**8 points**] Compute the best linear approximation $l(x)$ in the least-squares sense to the following data:

x	1	2	4	5	6
y	3	2	5	9	7

   Use $l(x)$ to estimate the value of y when $x = 3$.

4. [**8 points**] Estimate $\int_0^2 1/(1+x^3)\,dx$ by applying Simpson's rule with $h = 0.5$.

5. [**8 points**] Use Heun's method to estimate the solution of the differential equation
$$\dot{x} = x^2 - x/(1+t), \quad x(0) = 1;$$
   at $t = 0.5$ and $t = 1$.

Part II

During this part of the exam, you may *only* use the formula sheet provided, and the version of Matlab installed on the Department's computers. You may *not* use the internet, any other software, or your own laptop.

6. **[10 Points]** Compute the divided differences $f[x_i, \ldots, x_j]$, $i < j$ for the data:

x	0	0.5	1	1.5	2.0
y	1.000	0.959	0.841	0.665	0.455

   Write down the nested form of the interpolating quartic polynomial. Estimate the value of y when $x = 1.571$.

7. **[10 Points]** Compute the Romburg estimate $R_{4,4}$ to the definite integral

$$\int_0^5 \frac{\sin(x)}{x}\, dx.$$

   Write out the computed values of $R_{i,j}$ to 6 decimal places.

   Note: Use $R_{k,1}$ to denote the estimates obtained by the trapezoidal rule with 2^{k-1} subdivisions. Take $\sin(x)/x = 1$ when $x = 0$.

8. **[12 Points]** Consider the ordinary differential equation $\dot{x} = x - t^2$ with the initial conditions $x(0) = x_0 = 1.0$. Assume that an approximation $x_1 = 1.21859724$ has been computed to $x(0.2)$.

   (a) Use the two-step Adams-Bashforth method to compute an estimate $\tilde{x}_2$ to $x(0.4)$.

   (b) Apply the secant method starting with x_1 and $\tilde{x}_2$ to solve for x_2 in the fully implicit equation of the two-step Adams-Moulton method.

   (c) Continue your calculations to find a solution of the differential equation at $t = 1.0$. Show all your intermediate answers to 6 decimal places.

9. **[8 Points]** Consider the second-order ordinary differential equation

$$\ddot{x} + 12\dot{x} + 40x = \cos(\pi t/8); \qquad x(0) = 1, \quad \dot{x}(0) = 0.$$

   Show how to express this equation as a system of first-order equations, and write down a Matlab function `f(t,X)` expressing the right-hand side of this system.

   Compute the solution of the equation at time $t = 300$ using the builtin Matlab functions `ode45` and `ode15s`. Compare the properties of the two methods, both in terms of the number of steps taken, and the accuracy obtained. Which method is better for this problem? Explain your answer.

   Note: The exact solution has $x(300) = -0.002927$ to 6 decimal places.