DKE Numerical Mathematics
Resit June 2012

Answer all of the questions. Your highest grade on the homework assignments will be added to your score, and scores for one question in each part may be replaced by your other grades on the homework assignments.

Part I

During this part of the exam, you may *only* use the formula sheet provided, and an electronic calculator. You may *not* use a textbook or your own notes.

1. [**10 points**] Use the secant method to estimate the positive root of $x^3 - 5 = 0$ to within an accuracy of 10^{-8}, starting with $x_0 = 1$ and $x_1 = 2$.

2. [**10 points**] Compute the values $z_1 = s''(1)$ and $z_2 = s''(2)$ for the cubic spline $s(x)$ interpolating the data

x	0	1	2	3
y	1	2	5	7

   using the boundary conditions $s''(0) = s''(3) = 0$. Write down the polynomial $s_1(x)$ giving the values of s for $1 \leq x \leq 2$, and estimate the value of y when $x = 1.3$.

3. [**10 points**] Compute the best linear approximation $l(x)$ in the least-squares sense to the following data:

x	1.1	2.4	4.3	5.7	6.8	7.9
y	-1.2	-0.7	2.0	5.0	6.4	6.8

   Use $l(x)$ to estimate the value of y when $x = 3.0$.

4. [**10 points**] Estimate $\int_{-1}^{1} x/(x^3 + 2)\,dx$ by applying the composite Simpson rule with $h = 0.5$.

5. [**10 points**] Use the classical fourth-order Runge-Kutta method to estimate the solution of the differential equation

   $$\dot{x} = x - x^2/(1 + t), \quad x(0) = 1;$$

   at $t = 0.25$ and $t = 0.5$.

Part II

During this part of the exam, you may *only* use the formula sheet provided, and the version of Matlab installed on the Department's computers. You may *not* use the internet, any other software, or your own laptop.

6. **[10 Points]** Compute the divided differences $f[x_i, \ldots, x_j]$, $i < j$ for the data:

x	-1.0	-0.5	0.5	1.0	1.5
y	-0.719	-0.359	0.559	0.844	1.019

   Write down the nested form of the interpolating quartic polynomial. Estimate the value of y when $x = 0.0$.

7. **[10 Points]** Compute the coefficients c_i of the Legendre polynomials of the best least-squares approximation $g(x)$ to the function $f(x) = \exp(x)$ on $[-1, +1]$ by polynomials of degree at most 2. Evaluate g at the point $x = 0.5$ and compare with the value of f at $x = 0.5$.
   Hint: If p is a polynomial, then integration by parts gives
   $$\int_{-1}^{+1} p(x) \exp(x) dx = [p(x) \exp(x)]_{-1}^{+1} - \int_{-1}^{+1} p'(x) \exp(x) dx.$$

8. **[10 Points]** Compute the Romburg estimate $R_{4,4}$ to the definite integral
   $$\int_2^6 \frac{1}{x}\, dx.$$
   Write out the computed values of $R_{i,j}$ to 6 decimal places.

9. **[10 Points]** Consider the ordinary differential equation $\dot{x} = x - 3/(tx + 1)$ with the initial conditions $x(0) = x_0 = 2.0$. Assume that an approximation $x_1 = 1.87732418$ has been computed to $x(0.2)$.
   (a) Use the two-step Adams-Bashforth method to compute an estimate $\tilde{x}_2$ to $x(0.4)$.
   (b) Use the two-step Adams-Moulton formula as a corrector compute an improved estimate x_2 to $x(0.4)$.
   (c) Continue your calculations to find a solution of the differential equation at $t = 1.0$. Show all your intermediate answers to 6 decimal places.

10. **[10 Points]** Consider the second-order ordinary differential equation
    $$\ddot{x} + 24\dot{x} + 169x = 30 \sin(t); \qquad x(0) = 1, \quad \dot{x}(0) = 0.$$
    Express this equation as a system of first-order equations, and write down a Matlab function `f(t,X)` expressing the right-hand side of this system.

    Compute the solution of the equation at time $t = 100$ using the builtin Matlab functions `ode45` and `ode15s`, and describe the resulting behaviour over the interval $[0, 100]$, paying careful attention to the interval $[0, 1]$. Compare the properties of the two methods, both in terms of the number of steps taken, and the accuracy obtained. Which method is better for this problem?

    Note: The exact solution has $x(100) = 0.13824666$ to 8 decimal places.