DKE Numerical Mathematics
Exam 27 May 2013

Part I

During this part of the exam, you may *only* use the formula sheet provided, and an electronic calculator. You may *not* use a textbook or your own notes.

1. **[6 points]** Use two iterations of the secant method to estimate the positive root of $x^2 - 5 = 0$, starting your iteration with $p_0 = 1$ and $p_1 = 2$.

2. **[10 Points]** Compute the divided differences $f[x_i, \ldots, x_j]$, $i < j$ for the data:

i	0	1	2	3
x_i	1.0	2.0	2.5	3.5
$f(x_i)$	0.83	0.66	0.45	0.21

   Write down the nested form of the interpolating cubic polynomial. Estimate the value of y when $x = 1.5$.

3. **[8 points]** Use the compound Simpson's rule with $h = 0.5$ to approximate

$$\int_1^3 \frac{\sin x}{x}\, dx\ .$$

   By comparing your answer with that obtained using the simple Simpson's rule ($h = 1.0$) or otherwise, estimate the error of your answer.

4. **[10 points]** When using the two-stage Adams-Moulton method to solve an initial-value problem, which method would you prefer to use for the bootstrapping step, Euler's method or Heun's third-order method? Give a reason for your answer. Use the two-stage Adams-Bashforth method with step size $h = 0.5$ as a predictor, and the two-stage Adams-Moulton method as a corrector to estimate the solution of the differential equation

$$\dot{y} = 1 - y^2/(2 + t)$$

   at $t = 1.5$ with initial value $y(0.0) = 1.00000$. You may assume an approximation $y(0.5) \approx 1.25000$ is obtained using a single step of Euler's method, and $y(0.5) \approx 1.22392$ is obtained using a single step of Heun's third-order method.

5. **[6 points]** Compute the best linear approximation $y = l(x)$ in the least-squares sense to the following data:

x	1	2	3	4	5
y	2.0	5.0	3.0	4.0	6.0

   Does your approximation suggest that y is positively or negatively correlated with x?

Part II

During this part of the exam, you may *only* use the formula sheet provided, and the version of Matlab installed on the Department's computers. You may *not* use the internet, any other software, or your own laptop.

6. **[12 points]** Consider the cubic spline $s(x)$ interpolating the data

x	0	1	2	3	4
y	2.1	4.7	3.8	8.1	12.9

   using the boundary conditions $s'(0) = 4.0$ and $s''(4) = 0.0$.

   Write down and solve a system of linear equations from which you can determine the coefficients c_j of the quadratic terms. Determine the polynomial giving the values of s for $2 \leq x \leq 3$. Estimate the value of y when $x = 2.8$.

7. **[8 Points]** Estimate the derivative of the function $f(x) = \exp(x)/(2 + x)$ at $x = 0$ using at least two numerical methods with different step-sizes h. Compare your answer with the exact value $f'(0) = 0.25$. Which method and choice of h is most accurate, and why?

8. **[8 Points]** Use the 4th-order Runge-Kutta method with step-size $h = 0.1$ to solve the initial value problem

$$\dot{y} = 2 - 10y + y^2 - \sin(\pi t); \quad y(0) = 0;$$

   at time $t = 6$. Write down your intermediate results for the first step. Describe briefly the behaviour of y over the time interval $[0, 6]$.

9. **[8 Points]** Compute the Fourier series of the 2π-periodic function defined by $f(x) = \sin(x^2/\pi - x)$ for $x \in [-\pi, +\pi]$, finding terms up to $\cos(nx), \sin(nx)$ for $n = 2$.

10. **[4 Points]** Consider the system of ordinary differential equations

$$\dot{s} = v; \quad \dot{v} = (1 - s^2)v - s + \cos(2\pi t); \qquad s(0) = 1.0, \quad v(0) = 0.0.$$

    Write down a Matlab function `f(t,x)` expressing the right-hand side of this system.

    Compute the solution of the equation up to time $t = 12$ using the builtin Matlab functions `ode45` and `ode15s`. Which method is most efficient for this problem? Describe briefly the solution over the time interval $[0, 12]$.