DKE Numerical Mathematics
Resit 25 June 2013

Part I

During this part of the exam, you may *only* use the formula sheet provided, and an electronic calculator. You may *not* use a textbook or your own notes.

1. [**6 points**] Use Newton's method to find a solution of $x^3 - 2x - 1 = 0$, starting your iteration with $p_0 = 1.6$, and finishing when you obtain an approximation for with you estimate the error to be at most 10^{-3}.

2. [**10 Points**] Compute the divided differences $f[x_i, \ldots, x_j]$, $i < j$ for the data:

i	0	1	2
x_i	0.0	-1.0	-1.5
$f(x_i)$	0.63	0.43	0.24

   Hence, or otherwise, find the interpolating polynomial. Use your interpolation to estimate $\int_0^1 f(x)\,dx$.

3. [**8 points**] Use Simpson's to approximate $\int_1^4 f(x)\,dx$ from the following data.

x	1.0	2.0	3.0	3.5	4.0
$f(x)$	0.68	1.48	3.35	5.09	7.80

   Give accurate approximations to $f'(1.0)$ and $f'(3.5)$.

   Note: The data points x are not equally-spaced.

4. [**8 points**] Compute the solution of the initial-value problem

$$\dot{y} = 2t - y^2, \quad y(0.0) = 1.000$$

   at time $t = 0.5$ using

   (i) two steps of the midpoint method, and

   (ii) one step of the 4^{th}-order Runge-Kutta method.

   Compare your results with the exact answer $y(0.5) = 0.86209242$ (8dp).

5. [**8 points**] Compute the trigonometric polynomial best approximating the data (x_j, y_j) below in the least-squares sense, computing terms up to $\cos(2x)$, $\sin(x)$. (Values of $\cos(x_j)$ are also given.)

j	0	1	2	3	4	5	6	7
x_j	$-\pi$	$-\frac{3}{4}\pi$	$-\frac{1}{2}\pi$	$-\frac{1}{4}\pi$	0	$\frac{1}{4}\pi$	$\frac{1}{2}\pi$	$\frac{3}{4}\pi$
y_j	0.08	-0.19	-0.51	-0.02	1.90	2.30	1.11	0.38
$\cos(x_j)$	-1	$-1/\sqrt{2}$	0	$1/\sqrt{2}$	1	$1/\sqrt{2}$	0	$-1/\sqrt{2}$

   Use your result to approximate y when $x = \pi/3$.

Part II

During this part of the exam, you may *only* use the formula sheet provided, and the version of Matlab installed on the Department's computers. You may *not* use the internet, any other software, or your own laptop.

6. **[6 points]** use Matlab to compte a cubic spline interpolating the tdata

x	0.0	0.1	0.3	0.7	0.9	1.0
y	0.943	0.973	1.000	0.918	0.812	0.745

   using the boundary conditions $s'(0) = 0.3$ and $s'(1) = -0.7$.

   Determine the polynomial giving the values of s for $0.7 \leq x \leq 0.9$. Estimate the value of y when $x = 0.8$.

7. **[8 Points]** Let $f(x) = 1/(1 + x^3)$. Which partition width h is necessary to approximate $\int_1^2 f(x)\,dx$ to an accuracy of 10^{-3} using the composite midpoint rule? Use this partition width to approximate the integral.

   Hint: You may assume $|f''(\xi)| \leq 0.9$ for $\xi \in [1, 2]$.

8. **[12 Points]** Briefly explain the differences between a multi-step method and a single-step method for solving an ordinary differential equation, stating at least one advantage of each.

   Use the 2-stage Adams-Bashforth method as a predicator, and the 2-stage Adams-Moulton method as a corrector, to solve the differential equation

$$\dot{y} = 3e^{-t} - y/4$$

   with initial condition $y(0) = 0.000000$ over the time interval $[0, 12]$. Use the bootstrap condition $y(0.25) \approx 0.642449$.

   Give details of the intermediate calculations for the first step, including function evaluations $f_i = f(t_i, x_i)$, your approximation $w_i \approx y(t_i)$ for $t_i = 0.5$, 0.75, 1.0, and your approximation to $y(12)$.

   Describe your solution qualitatively. What, roughly, is the maximum value attained by y?

9. **[10 Points]** Using the recurrence relation for evaluating the Legendre polynomials P_k, or otherwise, compute the values of $P_k(0.8)$ for $k = 0, \ldots, 5$, and verify that $P_6(0.8) = -0.391796$.

   Compute the coefficients of the Legendre polynomials up to degree 6 in the least-squares approximation of the function $f(x) = e^{-x^2}$ over the interval $[-1, +1]$. Write down the resulting approximating polynomial.

10. **[4 Points]** Consider the system of ordinary differential equations

$$\dot{s} = 10(v + s - s^3/3); \quad \dot{v} = -s + \sin(2\pi t); \qquad s(0) = 2.0, \quad v(0) = 1.0.$$

   Write down a Matlab function `f(t,x)` expressing the right-hand side of this system.

   Compute the solution of the equation up to time $t = 10$ using the builtin Matlab functions `ode45` and `ode15s`. Which method is most efficient for this problem? Describe briefly the solution over the time interval $[0, 10]$.