DKE Numerical Mathematics
Exam 26 May 2014

Part I

During this part of the exam, you may *only* use the formula sheet provided, and an electronic calculator. You may *not* use a textbook or your own notes.

1. **[10 points]** Use Newton's method starting at $p_0 = 2$ to compute the positive solution of $x^3 - 4x - 2 = 0$ to an accuracy of 0.05. Give estimates for the absolute and relative errors of your answer.

2. **[10 Points]** Compute the divided differences $f[x_i, \ldots, x_j]$, $0 \leq i \leq j \leq 3$ for the data:

i	0	1	2	3
x_i	0.4	0.8	1.6	2.8
$f(x_i)$	0.37	0.65	0.98	0.72

   Write down the nested form of the interpolating cubic polynomial. Estimate the value of y when $x = 2.0$.

3. **[10 points]** Use the compound Simpson's rule with $h = 0.5$ to approximate

$$\int_0^2 \exp(-x^2)\, dx\ .$$

   By comparing your answer with that obtained using the simple Simpson's rule ($h = 1.0$) or otherwise, estimate the error of your answer.

4. **[10 points]** Compute the solution of the initial-value problem

$$\dot{y} = t - y^2, \quad y(0.0) = 1.5000$$

   at time $t = 0.5$ using one step of the 4^{th}-order Runge-Kutta method.

   Compare your results with the exact answer $y(0.5) = 0.94946720$ (8dp). What would you expect the error to be if you were to use *two* steps of the method to estimate $y(0.5)$?

5. **[10 points]** Compute the trigonometric polynomial best approximating the data (x_j, y_j) below in the least-squares sense, computing terms up to $\cos(2x)$, $\sin(x)$. (Values of $\cos(x_j)$ are also given; note $1/\sqrt{2} = 0.707107$ (6dp).)

j	0	1	2	3	4	5	6	7
x_j	$-\pi$	$-\frac{3}{4}\pi$	$-\frac{1}{2}\pi$	$-\frac{1}{4}\pi$	0	$\frac{1}{4}\pi$	$\frac{1}{2}\pi$	$\frac{3}{4}\pi$
y_j	-0.88	-0.50	0.76	0.98	0.88	0.86	0.76	0.18
$\cos(x_j)$	-1	$-1/\sqrt{2}$	0	$1/\sqrt{2}$	1	$1/\sqrt{2}$	0	$-1/\sqrt{2}$

   Use your result to approximate y when $x = \pi/3$; note $\sin(\pi/3) = 0.866025$ (6dp).

Part II

During this part of the exam, you may *only* use the formula sheet provided, and the version of Matlab installed on the Department's computers. You may *not* use the internet, any other software, or your own laptop.

6. **[15 points]** Consider the cubic spline $s(x)$ interpolating the data

x	0	1	2	3	4
y	0.48	1.99	2.60	2.65	3.02

using the boundary conditions $s''(0) = 0.0$ and $s'(4) = 0.8$.

Write down and solve a system of linear equations from which you can determine the coefficients c_j of the quadratic terms. Determine the polynomial giving the values of s for $1 \le x \le 2$. Estimate the value of y when $x = 1.4$.

7. **[10 points]** The data below comes from an experiment in which y is expected to be a quadratic function of x, but which is contaminated by noise.

x_i	1	2	3	4	5	6
y_i	2.7	-1.2	1.1	2.1	4.2	11.4

Using Matlab's `polyfit` command, or otherwise, compute the quadratic (degree 2) approximation to y in the least-squares sense. Plot the data and sketch the resulting graph.

Use your approximation to estimate dy/dx when x is equal to 4. Why would you *not* use a standard method, such as the three-point centred formula, in this case?

8. **[5 points]**

A good uniform approximation to the function $f(x) = \exp(-2x^2)$ over the interval $[-1, +1]$ is

$$g(x) = c_0T_0(x) + c_2T_2(x) + c_4T_4(x) + c_6T_6(x)$$

where $T_k(x)$ is the k^{th} Chebyshev polynomial, and the coefficients c_k are

$$c_0 = 0.2329,\ c_2 = -0.2079,\ c_4 = 0.0499,\ c_6 = -0.0082.$$

Evaluate $g(0.4)$ using the recurrence relation for the Chebyshev functions T_k, and compare with $f(0.4) = 0.7261(4\text{dp})$.

9. **[5 Points]**

Consider the function

$$f_{\mu,\sigma}(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$$

for $\mu = 1.1$ and $\sigma = 0.001$.

Evaluate $f_{\mu,\sigma}(\mu)$ and sketch a graph of $f_{\mu,\sigma}$ by hand. Compare your sketch with the output given by Matlab over the interval $[-5, +5]$. Using built-in Matlab commands, compute $\int_{-5}^{+5} f_{\mu,\sigma}(x)\,dx$, and compare withn the exact answer 1.0000(4dp).

10. **[10 Points]**

Consider the *Airy* differential equation

$$\frac{d^2x}{dt^2} = tx.$$

Show how to express this as a system of first order differential equations in Matlab.

For the initial condition $x(-8) = 1$, $\dot{x}(-8) = 0$ [i.e. $x = 1$ and $dx/dt = 0$ when $t = -8$], describe the solution over the interval $[-8, 2]$. What do you think happens as $t \to \pm\infty$?

A solution $x(t)$ is *bounded* if $\lim_{t\to\infty} x(t) = 0$. Suppose $\dot{x}(0) = -1$. Estimate the value of $x(0)$ so that the solution is bounded.