DKE Numerical Mathematics
Exam Wednesday 3 June 2015

Part I

During this part of the exam, you may *only* use the formula sheet provided, and an electronic calculator. You may *not* use a textbook or your own notes.

1. **[10 points]** The polynomial $p(x) = (x-2)(x-3)(x-4)$ expands to $p(x) = x^3 - 9x^2 + 26x - 24$.

   Write down the nested (Horner) form of p. Use all three forms of the polynomial to evaluate $p(3.47)$, rounding to three significant figures (3sf) after *every* arithmetic operation. Give the absolute and relative errors, given that the exact value of $p(3.47)$ is -0.366177. Which method performs best, and why? Explain briefly the relevance of your results to double-precision floating-point arithmetic.

2. **[12 points]** Compute the coefficients c_0, c_1, c_2 of the cubic spline s interpolating the data

i	0	1	2
x_i	0.0	0.5	1.0
y_i	0.5	1.2	1.3

   using boundary conditions $s'(0.0) = 1.0$ and $s''(1.0) = 0.0$. Write down the polynomial s_1 valid for $x \in [0.5, 1.0]$.

3. **[8 points]** Use the composite midpoint rule with $h = 0.5$ to approximate

   $$\int_1^3 \exp(-x^2)\, dx\ .$$

   Use the fact that $\sup_{\xi\in[1,3]} |f''(\xi)| \leq 1$ to find a bound on the error of your approximation.

4. **[12 points]** Compute the solution of the initial-value problem

   $$\dot{y} = t - y^2, \quad y(0.0) = 2.000.$$

   at time $t = 1.5$ with a step-size of 0.5 using the Adams-Bashforth two-stage method as a predictor and the Adams-Moulton two-stage method as a corrector. Bootstrap your method using an appropriate single-step method, justifying your choice.

5. **[8 points]** Compute the linear least-squares approximation to the following data.

x	0	2	4	7	10
y	0.9	0.4	0.1	0.3	1.4

   Plot the data and sketch your approximating function. Do you think that a linear approximation is appropriate?

Part II

During this part of the exam, you may *only* use the formula sheet provided, and the version of Matlab installed on the Department's computers. You may *not* use the internet, any other software, or your own laptop.

6. **[10 points]** Use the secant method starting with $p_0 = 1$ and $p_1 = 2$ to find a solution of $e^x - x - 5 = 0$ in the interval $[1, 2]$ to an accuracy of 0.001. Give estimates for the absolute and relative errors of your answer.

7. **[15 Points]** Compute the divided differences $f[x_i, \dots, x_j]$, $i < j$ for the data:

i	1	2	3	4	5
x_i	0.3	0.8	1.1	2.0	2.5
$f(x_i)$	1.51	0.82	0.60	0.30	0.22

   Hence, or otherwise, find the interpolating polynomial, writing your answer in nested form. Use your interpolation to estimate $f(x)$ when $x = 1.0$.

8. **[10 points]** Use the adaptive Simpson's rule to compute

$$\int_0^3 \exp(x^2)\,dx$$

   to an estimated accuracy of 0.1.

9. **[5 points]** A good uniform approximation to the function $f(x) = \tan^{-1} x$ over the interval $[-1, +1]$ is

$$g(x) = c_1 T_1(x) + c_3 T_3(x) + c_5 T_5(x)$$

   where $T_k(x)$ is the k^{th} Chebyshev polynomial, and the coefficients c_k are

$$c_1 = 0.82843,\ c_3 = -0.04748,\ c_5 = 0.00488.$$

   Evaluate $g(0.4)$ using the recurrence relation for the Chebyshev functions T_k, and compare with the exact value of $f(0.4)$.

10. **[10 Points]** Consider the forced Van der Pol differential equation

$$\frac{d^2x}{dt^2} + \mu(x^2 - 1)\frac{dx}{dt} + x = A\sin(\omega t).$$

   Show how to express this as a system of first order differential equations in Matlab.

   For the parameters $\mu = 8.53$, $A = 1.2$ and $\omega = \pi/5$ and initial condition $x(0) = -2$, $\dot{x}(0) = 0$ [i.e. $x = -2$ and $dx/dt = 0$ when $t = 0$], compute the solution over the interval $[0, 50]$ using Matlab's builtin `ode45` solver, giving the values of x and $\dot{x}$ when $t = 10, 20, 30, 40, 50$. Describe qualitatively the solution over the time interval $[0, 50]$.