DKE Numerical Mathematics
Exam Wednesday 1 June 2016

Part I

During this part of the exam, you may *only* use the formula sheet provided, and an electronic calculator. You may *not* use a textbook or your own notes.

Part II

During this part of the exam, you may *only* use the formula sheet provided, and the version of Matlab installed on the Department's computers. You may *not* use the internet, any other software, or your own laptop.

1. **[10 points]** Use the secant method to estimate the root of $e^x - 5x = 0$ lying in $[2, 3]$ to within an accuracy of 0.1, starting with $p_0 = 2$ and $p_1 = 3$.

   Solution:
   [2] Secant rule $p_{n+1} = p_n - \frac{p_n - p_{n-1}}{f(p_n) - f(p_{n-1})} f(p_n)$.
   [1] $p_0 = 2$, $f(p_0) = -2.6109$; $p_1 = 3$, $f(p_2) = 5.0855$.
   [2] $p_2 = 3.0000 - (3.0000 - 2.0000)/(5.0855 - -2.6109) \times 5.0855 = 3.0000 - 0.12993 \times 5.0833 = 3.3333 - 0.66076 = 2.3392$.
   [2] $f(p_2) = -1.3229$, $p_3 = 2.4756$. $f(p_3) = -0.4889$, $p_4 = 2.5556$.
   [1] Stop if $|p_n - p_{n-1}| < e$.
   [1] $e_4 = |p_3 - p_4| = 0.08 < 0.1$.
   [1] $x \approx p_4 = 2.6 \pm 0.1$.

   Common mistakes:
   The termination condition is $|p_n - p_{n-1}| < \epsilon = 0.1$. This is usually sufficient to have $|p_n - p_*| < \epsilon$, where p is the root.

2. **[14 points]** Compute the coefficients c_0, c_1 of the cubic spline s interpolating the data

i	0	1	2
x_i	0.5	1.0	1.5
y_i	0.7	2.3	3.1

using boundary conditions $b_0 = s'(0.5) = 2.0$ and $c_2 = \frac{1}{2}s''(1.5) = 2.2$. Write down the polynomial s_1 valid for $x \in [1.0, 1.5]$. Use your result to approximate y when $x = 1.2$.

Solution:
[**1**] Equally-spaced nodes $h = 0.5$.
[**2**] Clamped left boundary $2c_0+c_1 = 3/h^2(a_0-a_1-b_0h) = (3/0.5^2)\times(2.3-0.7-2.0\times0.5) = 12 \times 0.6 = 7.2$.
(Equivalently, $2hc_0 + hc_1 = 3/h((a_0 - a_1)/h - b_0)$ so $c_0 + \frac{1}{2}c_1 = 3.6$.)
[**2**] Knot condition $c_0 + 4c_1 + c_2 = 3/h^2(a_2 - 2a_1 + a_0) = (3/0.5^2) \times (3.1 - 2 \times 2.3 + 0.7) = 12 \times (-0.8) = -9.6$.
[**1**] Right boundary condition $c_2 = 2.2$.
[**2**] Eliminate c_2, $c_0 + 4c_1 = -11.8$. Eliminate c_0, $c_1 = (2 \times (-11.8) - 7.2)/7 = -30.8/7 = -4.4$, $c_0 = (7.2 - c_1)/2 = (7.2 - (-4.4))/2 = 11.6/2 = 5.8$.
[**3**] Compute $b_1 = (a_2-a_1)/h-h/3(2c_1+c_2) = (3.1-2.3)/0.5-0.5/3\times(2\times(-4.4)+2.2) = 0.8 \times 2.0 - (-6.6)/6 = 2.7$.
Compute $d_1 = (c_2 - c_1)/3/h = (2.2 - (-4.4))/3/0.5 = 6.6 \times 2.0/3 = 4.4$. [**3**] Polynomial $s_1 = a_1+b_1(x-x_1)+c_1(x-x_1)^2+d_1(x-x_1)^2 = 2.3+2.7\times(x-1.0)-4.4\times(x-1.0)^2-4.4\times(x-1.0)^3$. $s_1(1.2) = 2.3+2.7\times0.2-4.4\times0.2^2+2.7\times0.2^3 = 2.3+0.54-0.176+0.0352 = 2.6992 = 2.7(1dp)$.

3. [**8 points**] Estimate $\int_1^4 f(x)dx$ by applying Simpson's rule to the following data:

x	0.0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0
$f(x)$	0.500	0.471	0.333	0.186	0.100	0.057	0.034	0.022	0.015

Solution:
[**1**] $a = 1$, $b = 4$.
[**1**] $n = 6$, $h = (b - a)/n = 0.5$
[**2**] $S_6 = (b - a)/(3n)(f(a) + 4f(a + h) + 2f(a + 2h) + 4f(a + 3h) + \cdots + 2f(b - 2h) + 4f(b - h) + f(b))$
[**3**] $S_6 = \frac{4-1}{3\times 6}(0.333 + 4 \times 0.186 + 2 \times 0.100 + 4 \times 0.057 + 2 \times 0.034 + 4 \times 0.022 + 0.015) = \frac{1}{6}(0.333 + 0.744 + 0.200 + 0.228 + 0.068 + 0.088 + 0.015) = \frac{1}{6}1.676 = 0.27933\cdots$
[**1**] $I \approx 0.279 \approx 0.28$.

Common mistakes:
The data is given to $3dp$, so the integral can only be correct to at most 3dp. Give at most 3dp in your answer!

4. [**8 Points**] If you were to compute the solution of the differential equation $\dot{x} = \cos(x) - t$ starting at $t = 1$, $x = 2$ using the adaptive Bogaki-Shampine method with an error of tolernce of 10^{-4} per unit time, would a step size of 0.1 be adequate?

Solution:
[**2**] $k_1 = hf(t,x)$, $k_2 = hf(t+\frac{1}{2}h, x+\frac{1}{2}k_1)$, $k3 = hf(t+\frac{3}{4}h, x+\frac{3}{4}k_2)$, $k = (2k_1+3k_2+4k_3)/9$, $k_4 = hf(t+h, x+k)$, $\kappa = (7k_1 + 6k_2 + 8k_3 + 3k_4)/24$. $q = (\epsilon h/|\kappa - k|)^{1/2}$.
[**1**] Accept if $q \geq 1$ i.e. $|\kappa - k| < \epsilon h$.
[**1**] $t = 1, x = 2, h = 0.1$
[**3**]

$$
\begin{aligned}
k_1 &= 0.1f(1,2) = 0.1 \times (-1.4161468) = -0.14161468;. \\
k_2 &= 0.1f(1.05, 1.92919266) = -0.14007729; \\
k_3 &= 0.1f(1.075, 1.89494203) = -0.13934991; \\
k &= -0.14009566; \\
k_4 &= 0.1f(1.1, 1.85990435) = -0.13850974; \\
\kappa &= -0.14008729;
\end{aligned}
$$

$$k - \kappa = -8.36\times 10^{-6}, \quad \epsilon h = 1\times 10^{-5}$$

[**1**] $q = 1.09$ accept.

5. **[10 points]** Compute the trigonometric polynomial best approximating the data (x_j, y_j) below in the least-squares sense, computing terms up to $\cos(2x)$, $\sin(x)$.

j	-3	-2	-1	0	1	2
x_j	$-\pi$	$-\frac{2}{3}\pi$	$-\frac{1}{3}\pi$	0	$\frac{1}{3}\pi$	$\frac{2}{3}\pi$
y_j	-0.50	0.76	0.98	0.88	0.86	0.18

Use your result to approximate y when $x = \pi/4$.

Note: $\cos(\pi/3) = 1/2$, $\sin(\pi/3) = \sqrt{3}/2 \approx 0.8660$, $\cos(\pi/4) = \sin(\pi/4) = 1/\sqrt{2} \approx 0.7071$.

Solution:

[2] $m = 3$, $a_k = \frac{1}{3}\sum_{j=-3}^{2} y_j \cos(kx_j)$, $b_k = \frac{1}{3}\sum_{j=-3}^{2} y_j \sin(kx_j)$.

[1] $a_0 = (-0.50 + 0.76 + 0.98 + 0.88 + 0.86 + 0.18)/3 = 3.16/3 = 1.0533$.

[4] $a_1 = \big(1\times(-0.50) + \frac{-1}{2}\times 0.76 + \frac{1}{2}\times 0.98 + \times 0.88 + \frac{1}{2}\times 0.86 + \frac{-1}{2}\times 0.18\big)/3$
$= (-0.50 - 0.38 + 0.48 + 0.88 + 0.43 - 0.09)/3 = 1.83/3 = 0.6100.$

$a_2 = \big(1\times(-0.50) + \frac{-1}{2}\times 0.76 + \frac{-1}{2}\times 0.98 + 1\times 0.88 + \frac{-1}{2}\times 0.86 + \frac{-1}{2}\times 0.18)/3$
$= (-0.50 - 0.38 - 0.49 + 0.88 - 0.43 - 0.09\big)/3 = -1.01/3 = 0.3367.$

$b_1 = (0\times(-0.50) + \frac{-\sqrt{3}}{2}\times 0.76 + \frac{-\sqrt{3}}{2}\times 0.98 + 0\times 0.88 + \frac{\sqrt{3}}{2}\times 0.86 + \frac{\sqrt{3}}{2}\times 0.18)/3$
$= (0 - 0.6582 - 0.8487 + 0 + 0.7448 + 0.1559)/3 = -0.6062/3 = -0.2021.$

[3] $s(x) = a_0/2 + a_1\cos(x) + a_2\cos(2x) + b_1\sin(x)$
$= 0.5267 + 0.6100\cos(x) - 0.3367\cos(2x) - 0.2021\sin(x).$

$x = \pi/4$ gives $\cos(x) = \sin(x) = 1/\sqrt{2} \approx 0.7071$, $\cos(2x) = 0$,
$s(2) = 0.5267 + 0.6100\times 0.7071 - 0.3367\times 0 - 0.2021\times 0.7071 = 0.5267 + 0.4313 - 0.1429 =$
$-0.8151 = 0.82$ (2dp).

6. **[10 Points]** Compute the divided differences $f[x_i, \ldots, x_j]$, $i < j$ for the data:

x	1.0	0.5	1.5	0.0	2.0
y	0.688	0.730	0.570	0.499	0.414

Write down the nested form of the interpolating quartic polynomial. Estimate the value of y when $x = 0.8$.

Solution:

[2] Divided differences $f[x_i] = f(x_i)$,
$f[x_i, \ldots, x_j] = (f[x_{i+1}, \ldots, x_j] - f[x_i, \ldots, x_{j-1}])/(x_j - x_i)$.

[5]
$$f[x_0, x_1] = \tfrac{f[x_1]-f[x_0]}{x_1-x_0} = \tfrac{y_1-y_0}{x_1-x_0} = \tfrac{0.730-0.688}{0.5-1.0} = \tfrac{0.042}{-0.5} = -0.0840.$$
$$f[x_1, x_2] = \tfrac{f[x_2]-f[x_1]}{x_2-x_1} = \tfrac{y_2-y_1}{x_2-x_1} = \tfrac{0.570-0.730}{1.5-0.5} = \tfrac{-0.160}{1.0} = -0.1600$$
$$f[x_0, x_1, x_2] = \tfrac{f[x_1,x_2]-f[x_0,x_1]}{x_2-x_0} = \tfrac{-0.1600-(-0.0840)}{1.5-1.0} = \tfrac{-0.0760}{0.5} = -0.1520$$

1.0	0.68800	−0.08400	−0.15200	0.26267	−0.10600
0.5	0.73000	−0.16000	−0.41467	0.15667	
1.5	0.57000	0.04733	−0.17967		
0.0	0.49900	−0.04250			
2.0	0.41400				

[3]
$$\begin{aligned} p(x) &= a_0 + (x - x_0)\big(a_1 + (x - x_1)\big(a_2 + (x - x_2)\big(a_3 + (x - x_3)a_4\big)\big)\big) \\ &= 0.688 + (x - 1.0)\big(-0.084 + (x - 0.5)\big(-0.152 + (x - 1.5)\big(0.262\bar{6} + (x - 0.0)(-0.106)\big)\big)\big) \\ p(0.8) &= 0.688 + (-0.2)\times\big(-0.084 + 0.3\times\big(-0.152 + (-0.7)\times\big(0.262\bar{6} + 0.8\times(-0.106)\big)\big)\big) \\ &= 0.721 \text{ (3dp)} \end{aligned}$$

Or ordered,

0.0	0.49900	0.46200	−0.54600	0.26267	−0.10600
0.5	0.73000	−0.08400	−0.15200	0.05067	
1.0	0.68800	−0.23600	−0.07600		
1.5	0.57000	−0.31200			
2.0	0.41400				

Common mistakes: There is no need to reorder the data!
Show your working! Write out the formula, a couple of You should also write out the divided differences.

7. **[10 Points]** Compute the Romburg estimate $R_{4,4}$ to the definite integral

$$\int_0^3 \sin(x^2)\,dx.$$

Write out the computed values of $R_{i,j}$ to 6 decimal places.

Note: Use $R_{k,1}$ to denote the estimates obtained by the trapezoidal rule with 2^{k-1} subdivisions.

Solution:

[3] Trapezoidal rule $h_n = \frac{b-a}{n}$, $T_n = h_n\big(\frac{1}{2}f(a) + \sum_{i=1}^{n-1} f(a+ih_n) + \frac{1}{2}f(b)\big)$ or
$T_1 = \frac{b-a}{2}\big(f(a)+f(b)\big)$; $T_{2n} = \frac{1}{2}T_n + \frac{1}{2}h_{2n}\sum_{i=1}^{n} f(a+(2i-1)h_{2n})$.
Romburg integals $R_{i,1} = T_{2^{i-1}}$, $R_{i,j} = R_{i,j-1} + (R_{i,j-1} - R_{i-1,j-1})/4^{j-1}$.

[4] $T_1 = \frac{3-0}{1}\big(\frac{1}{2}f(0) + \frac{1}{2}f(3)\big) = \frac{3}{1}\big(\frac{1}{2}\times 0.00000 + \frac{1}{2}\times 0.41212\big) = 0.61818.$

$$T_2 = \tfrac{3-0}{2}\big(\tfrac{1}{2}f(0.0)+f(1.5)+\tfrac{1}{2}f(3.0)\big)$$
$$= \tfrac{3}{2}\times(\tfrac{1}{2}\times 0.00000+0.77807+\tfrac{1}{2}\times 0.41212) = \tfrac{3}{8}\times 0.98413 = 1.47620.$$
$$T_4 = \tfrac{3-0}{4}\big(\tfrac{1}{2}f(0.0)+f(0.75)+f(1.5)+f(2.25)+\tfrac{1}{2}f(3.0)\big)$$
$$= \tfrac{3}{4}\times(\tfrac{1}{2}\times 0.00000+0.53330+0.77807-0.93933+\tfrac{1}{2}\times 0.41212) = \tfrac{3}{4}\times 0.57810 = 0.43358.$$
$$T_8 = \tfrac{3-0}{8}\big(\tfrac{1}{2}f(0.0)+f(0.375)+f(0.75)+f(1.125)+f(1.5)+f(1.875)+f(2.25)+f(2.625)+\tfrac{1}{2}f(3.0)\big)$$
$$= \tfrac{3}{8}\times(\tfrac{1}{2}\times 0.00000+0.14016+0.53330+0.95380+0.77807-0.36537-0.93933+0.57077+\tfrac{1}{2}\times 0.41212)$$
$$= \tfrac{3}{8}\times 0.87745 = 0.70404$$

[3]
$$R_{2,2} = R_{2,1} + (R_{2,1} - R_{1,1})/3 = 1.47620 + (1.47620 - 0.61818)/3 = 1.76221$$
$$R_{3,2} = R_{3,1} + (R_{3,1} - R_{2,1})/3 = 0.43358 + (0.43358 - 1.47620)/3 = 0.08603$$
$$R_{4,2} = R_{4,1} + (R_{4,1} - R_{3,1})/3 = 0.70404 + (0.70404 - 0.43358)/3 = 0.79420$$
$$R_{3,3} = R_{3,2} + (R_{3,2} - R_{2,2})/15 = 0.08603 + (0.08603 - 1.76221)/15 = -0.02571$$
$$R_{4,3} = R_{4,2} + (R_{4,2} - R_{3,2})/15 = 0.79420 + (0.79420 - 0.08603)/15 = 0.84141$$
$$R_{4,4} = R_{4,3} + (R_{4,3} - R_{3,3})/63 = 0.84141 + (0.84141 - -0.02571)/63 = 0.85518$$

$$R = \begin{array}{c|c|c|c}
0.618178 & & & \\ \hline
1.476199 & 1.762206 & & \\ \hline
0.433575 & 0.086034 & -0.025711 & \\ \hline
0.704045 & 0.794201 & 0.841413 & 0.855176
\end{array}$$

8. **[12 points]** When using the two-stage Adams-Moulton method to solve an initial-value problem, which method would you prefer to use for the bootstrapping step, Euler's method or Heun's third-order method? Give a reason for your answer.

   Use the two-stage Adams-Bashforth method as a predictor, and the two-stage Adams-Moulton method as a corrector with step size $h = 0.5$ to estimate the solution up to $t = 10.0$ of the differential equation

$$\dot{y} = \cos(y) - 1/(t + y); \quad y(0.0) = 1.000000$$

   What would happen to the error if you were to use a step-size of $h = 0.25$?

   You may assume an approximation $y(0.5) \approx 0.770151$ is obtained using a single step of Euler's method, and $y(0.5) \approx 0.871805$ is obtained using a single step of Heun's method. You should write out the first two steps in full, and give the values of y when $t = 2.0, 5.0, 10.0$.

   Solution:
   [2] Prefer Heun's method, as this is a third-order method, as is the two-stage Adams method, so take $y(0.5) \approx y_1 = -0.085521$.
   [2] Predictor $\hat{y}_{k+1} = y_k + \frac{1}{2}h\big(3f(t_k, y_k) - f(t_{k-1}, y_{k-1})\big)$.
   Corrector $y_{k+1} = y_k + \frac{1}{12}h\big(5f(t_{k+1}, \hat{y}_{k+1}) + 8f(t_k, y_k) - f(t_{k-1}, y_{k-1})\big)$.
   [4] $f(t_0, y_0) = f(0.0, 1.000000) = -0.459698$.

$$f(t_1, y_1) = f(0.5, 0.871804) = -0.085521.$$

$$\begin{aligned}\hat{y}_2 &= y_1 + \tfrac{1}{2}h\big(3f(t_1, y_1) - f(t_0, y_0)\big)\\ &= 0.871804 + \tfrac{1}{2}\times 0.5\times(3\times(-0.085521) - (-0.459698))\\ &= 0.871804 + 0.5\times(-0.101568) = 0.922589.\end{aligned}$$

$$f(t_2, \hat{y}_2) = f(1.0, 0.922589) = 0.083626.$$

$$\begin{aligned}y_2 &= y_1 + \tfrac{1}{12}h\big(5f(t_2, \hat{y}_2) + 8f(t_1, y_1) - f(t_0, y_0)\big)\\ &= 0.871804 + \tfrac{1}{12}\times 0.5\times(5\times 0.083626 + 8\times(-0.085521) - (-0.459698))\\ &= 0.871804 + 0.5\times 0.016139 = 0.879874.\end{aligned}$$

   [2] $f(t_2, y_2) = f(1.0, 0.879874) = 0.105298$.

$$\begin{aligned}\hat{y}_3 &= y_2 + \tfrac{1}{2}h\big(3f(t_2, y_2) - f(t_1, y_1)\big)\\ &= 0.879874 + \tfrac{1}{2}\times 0.5\times(3\times 0.1052976 - (-0.085521))\\ &= 0.879874 + 0.5\times(-0.101568) = 0.980228.\end{aligned}$$

$$f(t_3, \hat{y}_3) = f(1.5, 0.980228) = 0.153645.$$

$$\begin{aligned}y_3 &= y_2 + \tfrac{1}{12}h\big(5f(t_3, \hat{y}_3) + 8f(t_2, y_2) - f(t_1, y_1)\big)\\ &= 0.879874 + \tfrac{1}{12}\times 0.5\times(5\times 0.153645 + 8\times 0.1052976 - (-0.085521))\\ &= 0.879874 + 0.5\times 0.141344 = 0.950546.\end{aligned}$$

   [1] $y(2.0) = y_4 = 1.038586$, $y(5.0) \approx y_{10} = 1.362328$, $y(10.0) \approx y_{20} = 1.473689$ **[1]** Expect error to be multiplied by 1/8.

9. **[10 Points]** The coefficients c_k of the Legendre polynomials P_k up to degree 6 in the least-squares approximation of a function f over the interval $[-1,+1]$ are

k	0	1	2	3	4	5	6
c_j	-0.13	0.69	-0.02	-0.22	0.24	-0.01	-0.10

.

Using the recurrence relation for evaluating the Legendre polynomials P_k, compute the value of the approximation $g_6(x) = \sum_{k=0}^{6} c_k P_k(x)$ at $x = 0.4$.

Using the formula for $\int_{-1}^{+1} P_k(x)^2\,dx$, estimate the value of $\int_{-1}^{+1} f(x)^2\,dx$.

Solution:

[2] $P_0(x) = 1$, $P_1(x) = x$, $P_{k+1}(x) = \big((2k+1)xP_k(x) - kP_{k-2}(x)\big)/(k+1)$.

[3] $P_0(0.4) = 1.$,

$$P_1(0.4) = 0.4,$$
$$P_2(0.4) = (3\times 0.4\times 0.4 - 1\times 1.)/2 = -0.26$$
$$P_2(0.4) = (5\times 0.4\times 0.26 - 2\times 0.4)/2 = -0.440$$
$$P_4(0.4) = (7\times 0.4\times -0.440 - 3\times 0.26)/4 = -0.1130$$
$$P_5(0.4) = (9\times 0.4\times -0.113 - 4\times -0.440)/5 = 0.27064$$
$$P_6(0.4) = (11\times 0.4\times 0.27064 - 5\times -0.1130)/6 = 0.292636$$

[2] $g(x) = \sum_{k=0}^{6} c_k P_k(x) = c_0P_0(x) + c_1P_1(x) + \cdots + c_6P_6(x)$

$$\begin{aligned} g(0.4) &= -0.13\times 1.0 + 0.69\times 0.4 - 0.02\times(-0.26) - 0.22\times(-0.440) \\ &\quad + 0.24\times(-0.1130) - 0.01\times 0.27064 - 0.10\times 0.292636 \\ &= -0.13 + 0.276 + 0.0052 + 0.09680 - 0.027120 - 0.0027064 - 0.02926360 \\ &= 0.18891 = 0.19\,(2\text{dp}) \end{aligned}$$

[3] $$\begin{aligned} \int_{-1}^{+1} g(x)^2 &= \int_{-1}^{+1} \sum_{k=0}^{6} c_k^2 P_k(x)^2\,dx = \sum_{k=0}^{6} c_k^2 P_k(x)^2\,dx \\ &= \sum_{k=0}^{6} c_k^2 \int_{1}^{+1} P_k(x)^2\,dx = \sum_{k=0}^{6} \tfrac{2}{2k+1} c_k^2 \\ &= \tfrac{2}{1}\times(-0.13)^2 + \tfrac{2}{3}\times 0.69^2 + \tfrac{2}{5}\times(-0.02)^2 + \tfrac{2}{7}\times(-0.22)^2 \\ &\quad + \tfrac{2}{9}\times 0.24^2 + \tfrac{2}{11}\times(-0.01)^2 + \tfrac{2}{13}\times(-0.10)^2 \\ &= 0.00338 + 0.31740 + 0.00016 + 0.01383 + 0.01280 + 0.00002 + 0.00154 \\ &= 0.37955 = 0.38\,(2\text{dp}). \end{aligned}$$

10. [**8 Points**] Consider the second-order ordinary differential equations

$$\ddot{x} - (1 - x^2)\dot{x} + x = \cos(2\pi t); \qquad x(0) = 2.0, \quad \dot{x}(0) = 1.0.$$

Show how to express this equation as a system of first-order equations, and write down a Matlab function `f(t,y)` expressing the right-hand side of this system.

Compute the values of x and $\dot{x}$ when $t = 30$ using the builtin Matlab function `ode45`. Sketch the solution over the time interval $[0, 30]$ and describe briefly its qualitative properties.

Solution:
[**3**] Introduce new variable $v = \dot{x}$ and set $y = (y_1, y_2) = (x, v)$. Then $\dot{x} = v$ and $\dot{v} = \ddot{x} = \cos(2\pi t) - x + (1 - x^2)v$, so $\dot{y}_1 = y_2$ and $\dot{y}_2 = \cos(2\pi t) - y_1 + (1 - y_1^2)y_2$. In vector form,
$\dot{y} = \begin{pmatrix} c\dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} y_2 \\ \cos(2\pi t) - y_1 + (1 - y_1^2)y_2 \end{pmatrix}.$
[**2**] `f=@(t,y)[y(2);cos(2*pi*t)-y(1)+(1-y(1).^2).*y(2)]`, or
`g=(t,x,v)cos(2*pi*t)-x+(1-x.^2).*v, f=(t,y)[y(2);g(t,y(1),y(2))],`
[**1**] `t0=0; tf=30; y0=[2.0;1.0], sol=ode45(f,[t0,tf],y0)`. Find $x(30) = -1.677911$, $\dot{x}(30) = -1.697751$.
[**2**] Non-sinusoidal, almost periodic oscillations, period roughly 6.