This guide

- does **not** contain formulas and will **not** prepare you to actually solve questions
- does **not** contain graphs and animations showing the methods, these are really important and I honestly dont think its possible to understand the material properly without them (although our lecturer seems to think complicated gibberish symbols somehow makes more sense, but anyways), there is a lot of great material on youtube and online that do the job of visualising really well, and **often suitable resources will be linked**
- is **not** exhaustive, does not cover everything but **does** cover things which I think are important
- **might** contain mistakes (not that I know of, but probably)
- **does** contain an easy to understand summary that uses words as it someone to talking you through it, linking the different topics, (limited, see next point) reasoning about them
- think of it as a **guide/assistant/plan/steps** for learning the material but not a substitute, ideally you would be understanding the stuff from linked sources that visualize them and explain them well (or others) and reviewing them to make sure you covered everything from his slides and finally doing practice questions until you can do them with the formula sheet only
- generally speaking I try to choose short and to the point (but good) videos for most topics, except when something we should know from background or is really essential to understand deeply then I go for 3blue1brown videos
- feel free to reupload a new version of this document with better sources or as he adapts/changes the course

You need (in my opinion) for this course:

- really basic algebra
- calculus would help in the differential equations/differentiation/integration sections but its really easy to understand what is needed from the linked videos. Which is only the intuition for what is it we are calculating and what does it represent, actually how to calculate stuff is not needed, that's what we are learning here (at least the numerical way, in calculus you would(ve) learn(ed) the analytical way)
- this is a really long way to say all you need to pass this is the will/effort/discipline to learn a few formulas and how to apply them, bonus point for understanding the intuition of where they come from and how they work

You will almost certainly fail if this is your only resource and you only understand the words written here and what the videos without doing the extra work of practicing applying and solving the formulas

1. Computer arithmetic and errors
    a. This is easy points, fun introduction with Tom here: https://www.youtube.com/watch?v=PZRI1IfStY0&list=PL-xEHa0Tmq2w0vLOjmjFkbTUArmHFzbQV
    b. He cares about significant figures and how many you present a lot, so learn about significant figures if you are not comfortable with them
    c.
2. Finding roots of equations
    a. Residual vs error: error is what we always calculate difference between our solution and actual, residual is the actual function evaluation of our calculated root (should be a small number if we did things properly) so think of it like remaining (residual) to get to an actual root
    b. For a bracket [a, b] if a and b have different signs AND the function is continuous then there is a root in that interval
    c. Bisection method (https://www.youtube.com/watch?v=MlP_W-obuNg)
        i. Iterate over the bracket, finding the midpoint then using this point as a new edge of the bracket
        ii. When to stop: bracket is less than epsilon[1]
    d. Secant method (https://www.youtube.com/watch?v=_MfjXOLUnyw)
        i. Very similar to the bisection method
        ii. Instead of using the midpoint we use the point at which the secant (connect the two points of the bracket) intercepts the x axis
        iii. When to stop: difference between last two points is less than epsilon
    e. Newton's method (https://www.youtube.com/watch?v=E24zUEKqgwQ)
        i. Similar to secant method
        ii. Only one starting point is needed, not a bracket
        iii. When to stop: difference between last two points is less than epsilon
        iv. Uses the point at which the tangent intercepts the x axis to calculate the next approximation
        v. When to stop: difference between last two points is less than epsilon
    f. Comparison of all the methods (well, at least for once there is a good slide in these lectures)

[1] Epsilon: a small positive number to which we say if our calculated value is this far away from true value then it's good enough

Comparison of methods

Reliability

+ The bisection method always works.
– The Newton-Raphson method and the secant method may cycle or diverge.

Requirements

+ The bisection and secant methods only require function values.
– The Newton-Raphson method requires the derivative of the function.

Efficiency

– The bisection method converges only linearly, $\epsilon_{n+1} \sim \frac{1}{2}\epsilon_n$
+ The Newton-Raphson method converges superlinearly at rate $\epsilon_{n+1} \sim C\epsilon_n^2$, the secant method $\epsilon_{n+1} \sim C\epsilon_n^{1.6}$.
– Per evaluation of f or f', the Newton-Raphson method is only $O(\epsilon^{1.4})$, slower than the secant method $O(\epsilon^{1.6})$.

3. Differential equations - Initial value problems (https://www.youtube.com/watch?v=p_di4Zn4wz4&list=PLZHQObOWTQDNPOjrT6KVlfJuKtYTftqH6, it's a whole list if you are interested but the first one is more than enough)
    a. Basically it's about finding/plotting the equation which the ODE[2] describes
    b. This is usually quite difficult to do with analytical methods, in practice numerical methods are often used
    c. What does the method order mean (they are written next to each method below)?
        i. They describe the effect on accuracy when changing the step size parameters
        ii. First order: ½ the step size = ½ the error
        iii. Second order: ½ the step size = ¼ the error
    d. Eurlers method - 1st order (https://www.youtube.com/watch?v=q87L9R9v274): this is basically about predicting the next point (for a given delta, the smaller the delta the more accurate the approximation) on the function using the slope of the previous point (this is actually part of the Runge-kutta family but its first order so I put it seperate and all the second order together)
    e. Runge-kutta - 2nd order:
    A family of methods that modify/build on the euler's method by trying (and in most situations succeeding) in finding a better approximation of the slope to find the next point, again, euler uses the slope of the previous point
        i. Midpoint method: using the slope of the midpoint (https://www.youtube.com/watch?v=bkNgjgXdjlw)
        ii. Modified euler's (or trapezoid) method: 1) find the new point by euler's method 2) average the slopes of the previous point and the new point 3) calculate the new point again using this averaged slope (https://www.youtube.com/watch?v=A5ObpYPADPQ)
        iii. Ralston's method: (couldn't really find a good source): 1) find the point using normal euler 2) find the point using normal euler again but ⅔ of step size instead of all the way 3) average these two with the first one carrying the weight ¼ and second ¾ (no clue why we do any of this)

[2] ODE: ordinary differential equation, revise this

f. There are also some third and fourth order methods in the slides but these never showed up on an exam so I am not going to do them
g. Multi-step methods (https://www.youtube.com/watch?v=wX9L3G9TFBA):
    i. Runge-kutta/euler only use the info from the current point to calculate the next point, this is a little bit short-sighted, turns out there is this class of methods called multi-step methods that also uses information from previous/future calculations
    ii. Adams-Bashforth methods (covered in g):
        1. Past and present points, the higher the order the more points from the past you are using
        2. It seems to me that the magic is in the ratios and weights the points are awarded in the formulas
        3. Bootstrapping:
            a. Problem: basically sometimes in these methods we are using information from previous points, but we do not know the value of these points at the beginning as they are not given and we have not calculated them
            b. Solution: use a Runge-Kutta method (point e) for these
            c. Important: use a RK method of the same order or higher, for accuracy purposes obviously
    iii. Adams-Moulton methods (covered in g):
        1. Present and future points, the higher the order the more points from the future you are using
        2. These are more accurate then Adams-Bashforth for the same order
        3. The magic here seems to be in interpolating the next points using the derivative
    iv. Predictor-corrector approach (intuition you already have from the precious to points:
        1. A way to combine Adams-Bashforth and Adams-Moulton to get even better results
        2. Basically first make a rough estimate using Adams-Bashforth (predictor step), then put that estimate into Adams-Mouton, it will give a more refined result (corrector step)
        3. The corrector-step could be applied again and again to give a more accurate result but the gain of this repetition is usually not worth it (doing it the first time gives a noticeable gain though)
h. Multivariable differential equations: so far all the equations we solved have one input and one output, multivariable deals with multiple inputs and outputs, he does not really dive into them but tells us that most of the methods we learned for single can be carried forward, the section mostly focuses on how to formulate them into matlab so matlab can solve them

4. Taylor series (deep conceptual understanding: https://www.youtube.com/watch?v=3d6DsjIBzJ4, more applied: https://www.youtube.com/watch?v=LDBnS4c7YbA)
   Approximating a function (near a point) with some polynomials using derivatives of 1st 2nd 3rd order and so on, the more orders you include the more accurate
5. Polynomial Interpolation
   a. Similar to taylor series but instead of approximating a function close to a point we want to find a function which has certain results (y's) at specific points (x's)
   b. Lagrange polynomials (this guys is very enthusiastic https://www.youtube.com/watch?v=B67wkZ3DWc0)
      i. This method is not very accurate practically but is useful to understand the rationale behind the rest of the methods
      ii. I didn't really understand what the lagrange basis do and where they come from, as far as I am concerned it's quite simple, there is a formula that you apply for each x point that you have, this will give you an equation of degree matching the degree asked for in the question, you add up all of these formulas each multiplied by the corresponding y value
   c. Divided differences (https://www.youtube.com/watch?v=JJR3xALzKkw)
      i. Most accurate
      ii. It's mostly about estimating or predicting the value of a new data point and using this new information to make a better polynomial estimate
      iii. Very complicated and didn't it study it through, it seems like in most exams all he asked is for a basic application for it so just learn how to apply it
6. Differentiation
   a. Finite difference (https://www.youtube.com/watch?v=g3Xw1r7QGOE)
      i. This is a fancy name for something super basic (and shit accuracy), using the definition of differentiation which has this h thing that we take to be really small then we do the calculation and we get the answer
      ii. Generally the smaller the h the better, but at the same time, too small of an h leads to all sorts of computer arithmetic truncation errors
   b. First derivative
      i. Centered difference (https://www.youtube.com/watch?v=safjLBaOWuA)
         Basically, get two points, one to the left and one to the right of our target derivative, then find the slope of the line connecting these two points, that is usually quite close to the derivative of our target, do watch the video above, its really amazing
      ii. Forward/backward difference (the intuition is similar to centred difference)
         Centred difference causes a problem at the end point of the domain, for the first point you dont have a point before it, for the last point you dont have a point after it, the solution to this is to use two points infornt/behind respectively
      iii. Two types of centred/forward/backward difference:

            1. Three points (explained above)
            2. Five points, similar but instead of two points we take four points on each side
            Forward/backward formulas for five points is not in the formula sheet so I assume he doesnt care about those.
        iv. Important notes about the formula sheet for these;
            1. He has the error terms in there, the ones with Xi "ξ", you can just ignore them when doing the calculations, dont ask me why he puts it there, probably to confuse us.
            By the way, if you think its because he might want us to find the error, I have interesting news for you... there is a no way of calculating ξ, go figure
            2. He only provides the formula for forward difference and not backward difference, I assume this is because he wants you to give birth to it in exam, fortunately its easy and just involves switching the signs a bit
    c. Second derivative
    There is a similar formula to the first derivative centered difference that is derived in a similar way. He has brief slides on points/forwards/backwards but skips them in formula sheet so I will skip them
7. Integration (https://www.youtube.com/watch?v=rfG8ce4nNh0)
    a. Reimann sums (https://www.youtube.com/watch?v=CXCtqBIEZ7g) (this concept is demonstrated in the video above by 3b1b but he doesnt call them that)
    A fancy way for saying apprxoimating the area under a curve (integral) using partitions (boxes, squares, trapezoids, anything)
    b. Midpoint rules
        i. Single interval
        This is useless, I do not know why he included it, but essentially you assume that the graph is a rectangle, find the midpoint height and multiply by the width
        ii. Multiple intervales
        Same but just several smaller intervals are used
    c. Trapezoid
    Similar to midpoint rules but trapezoid instead of a rectangle, also could be used with single interval or multiple intervals
    d. Simpsons rule (https://www.youtube.com/watch?v=uc4xJsi99bk)
    Better than both midpoint and trapezoid, by using interpolation, so you actually finding the area of another equation that behaves similar to the our target equation
    e. Romburg (https://www.youtube.com/watch?v=HtkylnbJklE)
        i. Richardson Extrapolation, this is important to understand Romburg, its basically a way to combine estimates with different number of partitions of

the simpsons rule, the combined estimates yield to less errors. This is usually done in a chain, from many partitions to less and less partitions.

f. Adaptive integration
    i. This is about improving our error, but its not an entirely new technique, it just uses the techniques we have learned already to further optimise the results for divisions with high errors, either by changing the technique or using more divisions over a certain subinterval
    ii. But how do we know the error over an interval? We use two different techniques and see which have significant deviations
    iii. In practice, we have a certain error tolerance, we don't divide it equally as certain intervals might be larger or small than others so we account for that

8. Approximation
    a. Data approximation
        i. Criteria: least squares
           Sum of the differences between actual and calculated squared
           He presented others (for example weighing certain points more because they are more important) but this is the only one I think is used in the course, if you are interested check them out in the slides.
        ii. Linear Least Squares
            1. Fitting a bunch of points to a linear line
    b. Functions approximation
        i. Criteria: integral of the squared difference between the two functions, so how big of an area are we over/under
    c. Fourier Series and Transforms
       (https://www.youtube.com/watch?v=UKHBWzoOKsY, this is a nice "article" https://betterexplained.com/articles/an-interactive-guide-to-the-fourier-transform)
        i.