Probabilities & Statistics 2016

Exam

Information: You have 3 hours. Please write clearly your answer. When an explanation is required, please justify your answer as clearly as you can. When numerical values are involved, you do not need to make all the calculations to the end.
There are a total of **100 points**.

<u>Good Luck!!!</u>

Question 1 (40 points). Answer the following questions:

1. We keep rolling a 20-faced fair die again and again.
   - What is the expected number of rolls till we see a 10?
   - Explain why.
   - What is the expected number of rolls till we observe a 10 followed by an 11?
   - Explain why.
2. We flip a fair coin 5 times.
   - What is the probability that we will observe more Heads than Tails?
3. We have a pair of 4-sided dice. We assume that both dice are fair (i.e., each face/number of the die has equal probability of coming up). Calculate the following probabilities:
   (a) Probability that the sum of the numbers on the two rolls is even.
   (b) Probability that the second roll is equal to the first roll.
   (c) Probability that the first roll is larger than the second.
   (d) Probability that at least one roll is equal to 4.
4. We have a fair coin and we flip it three times independently. Consider the following two events associated with the three flips: $A = \{\text{more heads than tails}\}$ and $B = \{\text{Heads on 1st Flip}\}$.
   - Compute the probability $P(A|B)$.
5. We flip a fair coin twice, independently. Consider the following three events: $A = \{\text{1st flip is H}\}$, $B = \{\text{2nd flip is H}\}$ and $C = \{\text{the two flips have different results}\}$.
   - Are A, B independent events? Why?
   - Are A, C independent? What about B, C?
   - Are A, B, C independent? Explain your answer.
6. Can the expectation of a random variable be negative? What about its variance? Justify briefly your answer.
7. Let X be a random variable such that $E(X) = 5$ and $Var(X) = 2$.

- What is $E(X^2)$?

Question 2 (10 points). Let $X_1, \ldots, X_8$ and $Y_1, \ldots, Y_8$ be independent random variables, uniformly distributed in the interval $[0,1]$. Let

$$Z = \frac{\sum_{i=1}^{8} X_i + \sum_{i=1}^{8} Y_i}{8}.$$

1. Calculate $E[Z]$ and $Var[Z]$.

2. Calculate the probability $Pr(|Z - E[Z]| \leq 0.01)$.

Question 3 (15 points). Alice has a bag and inside the bag there are 4 dice. She knows that 2 of them are 6-sided, 1 of them is 8-sided and the last one is a 12-sided die. She chooses one of the dice from the bag uniformly at random, she rolls it and gets a 7.

1. Given the data available to Alice, compute the prior probabilities, the likelihood function, and the posterior probabilities for each die.

2. Given the fact that Alice rolled a 7, compute the probability that she will again obtain a 7 if she rolls the same die once more.

Question 4 (15 points). You have a set of data point that are drawn from a distribution which you know is Gaussian. You also know that the variance of the distribution is $\sigma^2 = 16$ but the mean (expectation) of the particular distribution is unknown. You want to set up a Hypothesis testing regarding the mean μ of the distribution as follows:

- **Null Hypothesis:** The mean μ of the distribution is 2. In other words, the sampled data points follow $\mathcal{N}(2, 4^2)$.

- **Alternative Hypothesis:** The mean of the normal distribution is *not* 2.

- **Test Statistics:** The Standardize Sample Mean z.

- **Significance Level:** Set to be $\alpha = 0.05$.

You set up an experiment and you sample 16 experimental points. You calculate their sample mean and you find that it is $\bar{x} = 1.75$

1. Give the formula from which the sample mean was calculated.

2. Find the *Rejection Region* of your Hypothesis Testing. Draw carefully the graph showing:

   (a) The rejection region, and

   (b) The Null Distribution.

3. Find the z value. Put is in the figure your drawn in the above part.

4. Calculate the p-value for the above data. Interpret this value and decide whether to reject H_0 and adopt H_A or not.

Question 5 (20 points). A source A wants to communicate with a destination B in a networking environment. A transmits a message, which is a sequence of *binary* symbols: Each symbol is either a 1 with probability $p \in (0,1)$ or a 0 with probability $1-p$. Symbol 1 is received incorrectly (as 0) by B with probability ϵ_1 and symbol 0 is received incorrectly (as 1) by B with probability ϵ_0. Each error in transmission is independent of any other errors.

1. What is the probability that the ith symbol is received correctly?

2. What is the probability that the sequence 1001 is received correctly by B?

3. Consider the following simple technique to improve reliability: Each symbol is transmitted by A 3 times (so if A wants to send a zero she sends 000) and is decoded by B by a simple majority rule. I.e., B decodes the three symbols as 0 if in the 3 symbols he receives, there are at least 2 0s. Otherwise he decodes the three symbols as 1.

   Calculate the probability that B correctly decodes a 0 (in other words, the probability that B will decode a 0 when A send him a 0).

4. Suppose that we use the above decoding scheme and suppose that B receives 101. What is the probability that the original symbol transmitted by A was 0?

Standard normal table of left tail probabilities.

z	$\Phi(z)$	z	$\Phi(z)$	z	$\Phi(z)$	z	$\Phi(z)$
-4.00	0.0000	-2.00	0.0228	0.00	0.5000	2.00	0.9772
-3.95	0.0000	-1.95	0.0256	0.05	0.5199	2.05	0.9798
-3.90	0.0000	-1.90	0.0287	0.10	0.5398	2.10	0.9821
-3.85	0.0001	-1.85	0.0322	0.15	0.5596	2.15	0.9842
-3.80	0.0001	-1.80	0.0359	0.20	0.5793	2.20	0.9861
-3.75	0.0001	-1.75	0.0401	0.25	0.5987	2.25	0.9878
-3.70	0.0001	-1.70	0.0446	0.30	0.6179	2.30	0.9893
-3.65	0.0001	-1.65	0.0495	0.35	0.6368	2.35	0.9906
-3.60	0.0002	-1.60	0.0548	0.40	0.6554	2.40	0.9918
-3.55	0.0002	-1.55	0.0606	0.45	0.6736	2.45	0.9929
-3.50	0.0002	-1.50	0.0668	0.50	0.6915	2.50	0.9938
-3.45	0.0003	-1.45	0.0735	0.55	0.7088	2.55	0.9946
-3.40	0.0003	-1.40	0.0808	0.60	0.7257	2.60	0.9953
-3.35	0.0004	-1.35	0.0885	0.65	0.7422	2.65	0.9960
-3.30	0.0005	-1.30	0.0968	0.70	0.7580	2.70	0.9965
-3.25	0.0006	-1.25	0.1056	0.75	0.7734	2.75	0.9970
-3.20	0.0007	-1.20	0.1151	0.80	0.7881	2.80	0.9974
-3.15	0.0008	-1.15	0.1251	0.85	0.8023	2.85	0.9978
-3.10	0.0010	-1.10	0.1357	0.90	0.8159	2.90	0.9981
-3.05	0.0011	-1.05	0.1469	0.95	0.8289	2.95	0.9984
-3.00	0.0013	-1.00	0.1587	1.00	0.8413	3.00	0.9987
-2.95	0.0016	-0.95	0.1711	1.05	0.8531	3.05	0.9989
-2.90	0.0019	-0.90	0.1841	1.10	0.8643	3.10	0.9990
-2.85	0.0022	-0.85	0.1977	1.15	0.8749	3.15	0.9992
-2.80	0.0026	-0.80	0.2119	1.20	0.8849	3.20	0.9993
-2.75	0.0030	-0.75	0.2266	1.25	0.8944	3.25	0.9994
-2.70	0.0035	-0.70	0.2420	1.30	0.9032	3.30	0.9995
-2.65	0.0040	-0.65	0.2578	1.35	0.9115	3.35	0.9996
-2.60	0.0047	-0.60	0.2743	1.40	0.9192	3.40	0.9997
-2.55	0.0054	-0.55	0.2912	1.45	0.9265	3.45	0.9997
-2.50	0.0062	-0.50	0.3085	1.50	0.9332	3.50	0.9998
-2.45	0.0071	-0.45	0.3264	1.55	0.9394	3.55	0.9998
-2.40	0.0082	-0.40	0.3446	1.60	0.9452	3.60	0.9998
-2.35	0.0094	-0.35	0.3632	1.65	0.9505	3.65	0.9999
-2.30	0.0107	-0.30	0.3821	1.70	0.9554	3.70	0.9999
-2.25	0.0122	-0.25	0.4013	1.75	0.9599	3.75	0.9999
-2.20	0.0139	-0.20	0.4207	1.80	0.9641	3.80	0.9999
-2.15	0.0158	-0.15	0.4404	1.85	0.9678	3.85	0.9999
-2.10	0.0179	-0.10	0.4602	1.90	0.9713	3.90	1.0000
-2.05	0.0202	-0.05	0.4801	1.95	0.9744	3.95	1.0000

$\Phi(z) = P(Z \leq z)$ for $\mathrm{N}(0, 1)$.

(Use interpolation to estimate z values to a 3rd decimal place.)

Figure 1: Table for the Standard Gaussian Probabilities