Maastricht University

Department of Data Science and Knowledge Engineering

Linear Programming 2018/2019
Exam Questions

— Do not turn this page before the official start of the exam! —

First Name, Surname: ____________________

Student ID: ____________________

Program: Bachelor Data Science and Knowledge Engineering

Course code: KEN2520

Examiner: Dr. Steven Kelk

Date/time: Thursday June 6$^{\text{th}}$ 2019, 09.00-12.00h.

Format: Closed book exam

Allowed aids: Pens, simple (non-programmable) calculator from the DKE-list of allowed calculators.

Instructions to students:

- The exam consists of 7 questions on 17 pages (excluding the 1 cover page(s)).
- Fill in your name and student ID number on each page, including the cover page.
- Answer every question at the reserved space below the questions. If you run out of space, continue on the back side, and if needed, use the extra blank pages.
- Ensure that you properly motivate your answers.
- Write in a readable way. Answers that cannot be read easily cannot be graded and may therefore lower your grade.
- You are not allowed to have a communication device within your reach, nor to wear or use a watch.
- You have to return all pages of the exam. You are not allowed to take any sheets, even blank, home.
- If you think a question is ambiguous, or even erroneous, and you cannot ask during the exam to clarify this, explain this in detail in the space reserved for the answer to the question.
- If you have not registered for the exam, your answers will not be graded, and thus handled as invalid.
- **Good luck!**

The following table will be filled by the examiners:

Question:	1	2	3	4	5	6	7	Total
Points:	15	15	15	15	15	12	13	100
Score:								

Question 1 (15 points)

In our warehouse we have 200 *red* flowers, 300 *blue* flowers and 100 *green* flowers. The longer the flowers stay in the warehouse, the less profit they can generate, because they get older and thus less attractive to customers. Specifically:

- Each red flower sold *today* at the market generates 4 euros of profit, each red flower sold *tomorrow* at the market generates 2 euros of profit.
- Each blue flower sold *today* at the market generates 2 euros of profit, each blue flower sold *tomorrow* at the market generates 1.5 euros of profit.
- Each green flower sold *today* at the market generates 3 euros of profit, each green flower sold *tomorrow* at the market generates 2 euros of profit.

Our goal is to decide how many flowers to sell on which day, such that profit is maximized, and such that the following additional constraints are also satisfied:

- Today we can sell at most 150 flowers. Tomorrow we can sell at most 250 flowers. (This is because tomorrow is a national holiday, and we expect more visitors to the market).
- Today, at least 30% of the flowers we sell should be red.
- Any blue flowers left in the warehouse after tomorrow, are simply thrown away, and do not generate any profit.
- Any red or green flowers left in the warehouse after tomorrow, are automatically sold at a heavy discount, generating 0.5 euros of profit per flower.

(You can assume that, if we decide to sell a given flower today or tomorrow at the market, that it will definitely be sold i.e. flowers are never returned to the warehouse.)

Formulate a linear program to solve this optimization problem. It is fine if your model potentially prescribes a fractional number of flowers for some purpose. Clearly define your variables, objective function and constraints. Note that you do *not* need to solve the linear program you write.

Question 2 (15 points)
Consider the following linear program:

$$\begin{aligned} \textit{Maximize} \quad & 4x_1 + 3x_2 + 3x_3 \\ \text{s.t.} \quad & -1x_1 + 2x_2 - 2x_3 \leq 40 \\ & 2x_1 - 2x_2 + 2x_3 \leq 30 \\ & -3x_1 + 3x_2 + 2x_3 \leq 25 \\ \text{and} \quad & x_1, x_2\, x_3 \geq 0. \end{aligned}$$

a. Solve the program using the Simplex method, starting from the usual initial solution (i.e. all decision variables are nonbasic). Show each Simplex tableau clearly. When you are finished, state explicitly the optimum value Z^* of the objective function and the values of the decision variables and the slack variables. **Hint:** If you do this correctly you will require at most **3** pivots and at optimality all decision and slack variables will be **integral**.

b. For the solution you found in part (a), explain which variables are basic and nonbasic, and give an interpretation of this in terms of the tightness of the functional constraints and non-negativity constraints of the original program.

Question 3 (15 points)

The following subquestions explore a number of miscellaneous topics. All the subquestions carry equal weight. A few lines per subquestion should be sufficient; long answers are *not* necessary!

a. A linear program can have exactly one of three states: (F)easible with finite optimum, feasible with (U)nbounded optimum, and (I)nfeasible. Explain briefly in words how the Simplex method detects which of these situations holds.

b. Suppose we have a linear program which has three non-negative decision variables x_1, x_2, x_3 and three functional constraints, each containing a single variable: $x_i \leq 1$, $i \in \{1, 2, 3\}$.

   1. How many basic feasible solutions does this program have?
   2. How many basic solutions does this program have?
   3. Does the linear program have any *degenerate* basic feasible solutions?

c. In the Simplex method we can recode $k \geq 1$ *unconstrained* decision variables (i.e. variables that are potentially negative) as $k + 1$ *non-negative* decision variables. Explain briefly how this is done.

Question 4 (15 points)
The following two subquestions are unrelated to each other.

a. Consider the following (primal) linear program:

$$\begin{array}{llll} \textit{Maximize} & Z = & 3x_1 - x_2 + x_3 & \\ \text{s.t.} & & x_1 - x_2 + 3x_3 & \leq 4 \\ & & 2x_1 + x_2 & \leq 12 \\ & & x_1 - x_2 - x_3 & \leq 7 \\ & & x_1, x_2, x_3 \geq 0. & \end{array}$$

Let y_i be the dual decision variable corresponding to the ith primal constraint. You are told that there is an optimal basic feasible solution for the *dual* of the above program with the following properties: y_1 and y_2 are greater than zero and the third dual constraint is not tight. Use this information to derive an optimal basic feasible solution for the original *primal* program.

b. Use the SOB method to write down the dual of the following (primal) linear program.

$$\begin{array}{rll} \textit{Maximize} & -8x_1 - 12x_2 & \\ \text{s.t.} & 4x_1 + 6x_2 & \geq 7 \\ \text{and} & x_1 \geq 0, & \\ & x_2 \leq 0. & \end{array}$$

Use the dual you constructed and the Strong Duality Theorem to argue that the optimum of the above program is -14.

Question 5 (15 points)

We have seen during the course that it is possible to describe the Simplex tableaux only in terms of values $\mathbf{b}$, $\mathbf{B}$, $\mathbf{c}$, $\mathbf{c}_B$, and $\mathbf{A}$:

$\mathbf{c}_B\mathbf{B}^{-1}\mathbf{A} - \mathbf{c}$	$\mathbf{c}_B\mathbf{B}^{-1}$	$\mathbf{c}_B\mathbf{B}^{-1}\mathbf{b}$
$\mathbf{B}^{-1}\mathbf{A}$	$\mathbf{B}^{-1}$	$\mathbf{B}^{-1}\mathbf{b}$

Consider the following linear program:

$$\begin{aligned} Maximize \quad & -2x_1 + 4x_2 + 3x_3 \\ \text{s.t.} \quad & 3x_1 + 1x_2 + 1x_3 \le 40 \\ & 1x_1 - 1x_2 + 2x_3 \le 10 \\ & -1x_1 + 1x_2 - 1x_3 \le 20 \\ & x_1, x_2, x_3 \ge 0. \end{aligned}$$

Let x_4, x_5 and x_6 denote the slack variables for the first, second and third constraints, respectively. Suppose that after some number of iterations of the simplex method, you are shown the incomplete simplex tableau below.

```
Bas|Eq|              Coefficient of              | Right
Var|No| Z|   X1    X2    X3    X4    X5    X6    |  side
___|__|__|_______________________________________|______
   |  |  |                                       |
 Z | 0| 1|    ?     ?     ?   0.5     ?   3.5    |   90
 X?| 1| 0|    ?     ?     ?   0.25    ?     ?    |   ?
 X?| 2| 0|    ?     ?     ?   0       ?     1    |   30
 X?| 3| 0|    0     ?     ?   0.25    ?     ?    |   ?
```

Fill in the missing numbers in the table. Explain your reasoning carefully. Do not just pivot mechanically: use analytical/algebraic arguments to justify your answers!

Question 6 (12 points)

Let $G = (V, E)$ be an undirected graph. Recall that a *vertex cover* of G is a subset $V' \subseteq V$ such that, for every edge $e = \{u, v\} \in E$, $V' \cap \{u, v\} \neq \emptyset$. In other words, every edge has at least one endpoint in V'. We are interested in constructing a vertex cover of *minimum* size. In this variant there are no weights on the vertices.

The natural Integer Linear Program (ILP) for this problem, which we have discussed throughout the course, has one binary variable x_u for each vertex u and one constraint per edge.

a. *(2 points)* Write down the natural ILP for this problem, and its LP-relaxation.

Consider a feasible solution x to the LP-relaxation described in (a). Let $V_0 \subseteq V$ be those vertices whose decision variables are 0 in x. Similarly, let $V_1 \subseteq V$ be those vertices whose decision variables are 1 in x. Let $V_F \subseteq V$ be all remaining vertices i.e. vertices whose decision variables are given a fractional value in x.

b. *(2 points)* Prove that, for a vertex $u \in V$, if $u \in V_F$ and $\{u, v\} \in E$, then $v \notin V_0$.

Suppose that G is *bipartite*. That is, V can be partitioned into two non-empty, non-overlapping subsets V_L, V_R such that each edge in E has one endpoint in V_L and one endpoint in V_R.

c. *(5 points)* Prove that, if we choose a sufficiently small constant $\epsilon > 0$, adding ϵ to all decision variables corresponding to vertices in $V_F \cap V_L$, and at the same time subtracting ϵ from all decision variables corresponding to vertices in $V_F \cap V_R$, yields a feasible solution.

d. *(3 points)* Using the insight from (c), prove that when G is bipartite every basic feasible solution to the LP-relaxation in (a) is integral (i.e. has no fractional values).

Question 7 (13 points)
Once we have finished executing the Simplex method we have a Simplex tableau corresponding to an optimal basic feasible solution. Suppose we then change the original linear program in some way. Sensitivity analysis helps us understand the new role of the previously optimal basis, without applying the Simplex algorithm from the very beginning again. We often commence sensitivity analysis by writing down the *revised* final Simplex tableau, where $\overline{\mathbf{A}}$, $\overline{\mathbf{b}}$, $\overline{\mathbf{c}}$ are the modified versions of $\mathbf{A}$, $\mathbf{b}$, $\mathbf{c}$ respectively:

$\mathbf{y}^*\overline{\mathbf{A}}-\overline{\mathbf{c}}$	$\mathbf{y}^*$	$\mathbf{y}^*\overline{\mathbf{b}}$
$\mathbf{S}^*\overline{\mathbf{A}}$	$\mathbf{S}^*$	$\mathbf{S}^*\overline{\mathbf{b}}$

Consider the following linear program. After introducing slacks x_4, x_5 and x_6 and solving with the Simplex method, we obtain the final Simplex tableau, also given below.

$$\begin{array}{rll} Maximize & 2x_1 + 5x_2 - 3x_3 & \\ \text{s.t.} & -1x_1 + 4x_2 + 5x_3 & \leq 50 \\ & -3x_1 + 2x_2 + 4x_3 & \leq 100 \\ & 2x_1 + 5x_2 - 2x_3 & \leq 30 \\ \text{and} & x_1, x_2, x_3 \geq 0. & \end{array}$$

```
Bas|Eq|                Coefficient of              | Right
Var|No| Z|   X1    X2     X3    X4     X5     X6   |  side
___|__|__|_________________________________________|______
   |  |  |                                         |
 Z | 0| 1|    0     0      1     0      0      1   |    30
X4| 1| 0| -2.6     0    6.6     1      0   -0.8   |    26
X5| 2| 0| -3.8     0    4.8     0      1   -0.4   |    88
X2| 3| 0|  0.4     1   -0.4     0      0    0.2   |     6
```

For each of the following questions we will always assume that this is the starting point for the proposed changes. *All subquestions carry approximately equal weight.*

a. Suppose we are only going to change the coefficient of x_1 in the objective function. Calculate the allowable increase and decrease in this coefficient (from its current value of 2) such that the previously optimal basic feasible solution remains optimal. **New:** Suppose we change this coefficient such that optimality is lost, and re-optimize. Will the new optimal basis *definitely* be $\{x_4, x_5, x_1\}$? Explain your answer very briefly.

b. Suppose we are only going to change the ≤ 30 part of the program. Calculate the allowable decrease and increase of this value such that the previously optimal basis remains feasible.

c. Suppose we are only going to change the coefficient of x_2 in the objective function. Calculate the allowable increase and decrease in this coefficient (from its current value of 5) such that the previously optimal basis remains optimal. What is the *maximum* value that the objective function can achieve in this allowable range?