Linear Programming 2019/2020
Exam Questions

— *Do not turn this page before the official start of the exam!* —

First Name, Surname: ______________________________

Student ID: ______________________________

Program: Bachelor Data Science and Knowledge Engineering

Course code: KEN2520

Examiner: Dr. Steven Kelk

Date/time: Tuesday June 2nd$^{\text{th}}$ 2020, 09.00-12.00h (actual exam time: 2 hours)

Format: Closed book exam

Allowed aids: Pens, simple (non-programmable) calculator from the DKE-list of allowed calculators.

Instructions to students:

- The exam consists of 6 questions on 6 pages (excluding the 1 cover page(s)).
- Ensure that you properly motivate your answers.
- Write in a readable way. Answers that cannot be read easily cannot be graded and may therefore lower your grade.
- You are not allowed to have a communication device within your reach, nor to wear or use a watch.
- If you think a question is ambiguous, or even erroneous, and you cannot ask during the exam to clarify this, explain this in detail in your answer to the question.
- If you have not registered for the exam, your answers will not be graded, and thus handled as invalid.
- **Good luck!**

The following table will be filled by the examiners:

Question:	1	2	3	4	5	6	Total
Points:	15	15	15	15	15	15	90
Score:							

Question 1 (15 points)

Sara and Menno have a small bakery shop that specializes in two types of cookies: plain and iced. They need to decide how many batches of each type of cookie to make in order to maximize their profit. One batch of their plain cookies requires 0.5 kg of cookie dough, and no icing, while one batch of their iced cookies requires 0.4 kg of cookie dough and 0.3 kg of icing. From experience Sara and Menno know that each batch of plain cookies requires 0.15 hours of preparation time, and each batch of iced cookies 0.25 hours of preparation time. Other relevant facts are: (1) They have in total 50 kg of cookie dough and 20 kg of icing. (2) They have room to bake at most 150 batches of cookies. (3) They can spend up to 16 hours on cookie preparation. Each batch of plain cookies costs them 5 euro to make, but can be sold in the shop for 6.50 euro. Each batch of iced cookies costs them 7 euro to make, but can be sold in the shop for 10.50 euro.

Formulate a linear program to determine how many batches of each cookie type Sara and Menno should produce, to maximize profit. Clearly define your variables, objective function and constraints. It is fine if your model potentially allows a fractional number of batches of cookies.

Note that you do *not* need to solve the linear program you write.

Question 2 (15 points)
Consider the following linear program:

$$\begin{array}{rl} \textit{Maximize} & 4x_1 - 2x_2 + 3x_3 \\ \text{s.t.} & 2x_1 + 1x_2 - 2x_3 \leq 40 \\ & -1x_1 - 2x_2 + 4x_3 \leq 15 \\ \text{and} & x_1, x_2, x_3 \geq 0. \end{array}$$

a. Solve the program using the Simplex method, starting from the usual initial solution (i.e. all decision variables are nonbasic). Show each Simplex tableau clearly. When you are finished, state explicitly the optimum value Z^* of the objective function and the values of the decision variables and the slack variables.

b. Write down the *shadow prices* of the solution you found in part (a).

Question 3 (15 points)

Consider the following linear program; we refer to this as the *primal* program.

$$\begin{array}{rl} \textit{Maximize} & 4x_1 + 3x_2 + 6x_3 \\ \text{s.t.} & 3x_1 + 1x_2 + 3x_3 \leq 30 \\ & 2x_1 + 2x_2 + 3x_3 \leq 40 \\ \text{and} & x_1, x_2, x_3 \geq 0. \end{array}$$

a. Consider the basic feasible solution B for the primal program in which $x_1 > 0$ and $x_2 > 0$. Construct the corresponding complementary basic solution for the *dual* program.

b. Use your answer to (a) to determine whether B is optimal for the primal program. If it is, state the optimum value Z^* it achieves. If it is not, state what the entering basic variable will be if one continues to optimize from this point.

c. Suppose we change "≤ 40" in the second functional constraint into "≥ 50". Suppose you use the big-M method to solve this modified program. Write down which variables will be basic in the initial (artificial) basic feasible solution, and their values. Also, write down which variables will be nonbasic at this point. (It is not necessary to set up the Simplex tableau, unless you find that useful).

Question 4 (15 points)

We have seen during the course that it is possible to describe the Simplex tableaux only in terms of values $\mathbf{b}$, $\mathbf{B}$, $\mathbf{c}$, $\mathbf{c}_B$, and $\mathbf{A}$:

$\mathbf{c}_B\mathbf{B}^{-1}\mathbf{A} - \mathbf{c}$	$\mathbf{c}_B\mathbf{B}^{-1}$	$\mathbf{c}_B\mathbf{B}^{-1}\mathbf{b}$
$\mathbf{B}^{-1}\mathbf{A}$	$\mathbf{B}^{-1}$	$\mathbf{B}^{-1}\mathbf{b}$

Consider the following linear program:

$$\begin{array}{rll} \textit{Maximize} & 5x_1 + 9x_2 + 7x_3 & \\ \text{s.t.} & 1x_1 + 3x_2 + 2x_3 & \leq 10 \\ & 3x_1 + 4x_2 + 2x_3 & \leq 12 \\ & 2x_1 + 1x_2 + 2x_3 & \leq 8 \\ & x_1, x_2, x_3 \geq 0. & \end{array}$$

Let x_4, x_5 and x_6 denote the slack variables for the first, second and third constraints, respectively. Suppose that after some number of iterations of the simplex method, you are shown the incomplete simplex tableau below. **You are also told that, at this point, x_2 is the only basic decision variable.**

```
Bas|Eq|             Coefficient of               | Right
Var|No| Z|  X1    X2    X3    X4    X5    X6    |  side
___|__|__|_______________________________________|______
   |  |  |                                       |
 Z | 0| 1|  ?     ?     ?     ?     ?       ?   |     ?
 X?| 1| 0|  ?     ?     ?     ?   -0.75     ?   |     1
 X2| 2| 0|  ?     ?     ?     ?    0.25     ?   |     3
 X?| 3| 0|  ?     ?     ?     ?   -0.25     ?   |     ?
```

Fill in the missing numbers in the table. Explain your reasoning carefully. Do not just pivot mechanically: use analytical/algebraic arguments to justify your answers!

Question 5 (15 points)

Let $G = (V, E)$ be an undirected graph. Recall that a *matching* of G is a subset $E' \subseteq E$ such that, for every $v \in V$, v is incident to (i.e. is an endpoint of) at most one edge in E'. We are interested in constructing a matching of *maximum* size. In this variant there are no weights on the edges.

The natural Integer Linear Program (ILP) for this problem has one binary variable x_e for each edge $e \in E$ and one functional constraint per vertex.

a. Write down the natural ILP for this problem, and its LP-relaxation.

b. In the LP-relaxation, constraints of the form $0 \leq x_e \leq 1$ can be replaced by $x_e \geq 0$ without altering the feasible region of the LP-relaxation. Briefly explain why.

c. Let P be the version of the LP-relaxation in which constraints $0 \leq x_e \leq 1$ have been replaced by $x_e \geq 0$. Write down the dual of P. Call this D.

d. Let C_n be the cycle graph on n vertices (so C_3 is a triangle, C_4 is a square, C_5 is a pentagon and so on). Using the dual D and any other arguments you think relevant, prove the following:

   for every **odd** $n \geq 3$, the size of the ILP optimum, divided by the size of the LP-relaxation optimum, is **exactly** $1 - \frac{1}{n}$ when applied to C_n.

Question 6 (15 points)

Once we have finished executing the Simplex method we have a Simplex tableau corresponding to an optimal basic feasible solution. Suppose we then change the original linear program in some way. Sensitivity analysis helps us understand the new role of the previously optimal basis, without applying the Simplex algorithm from the very beginning again. We often commence sensitivity analysis by writing down the *revised* final Simplex tableau, where $\overline{\mathbf{A}}$, $\overline{\mathbf{b}}$, $\overline{\mathbf{c}}$ are the modified versions of $\mathbf{A}$, $\mathbf{b}$, $\mathbf{c}$ respectively:

$\mathbf{y}^*\overline{\mathbf{A}}-\overline{\mathbf{c}}$	$\mathbf{y}^*$	$\mathbf{y}^*\overline{\mathbf{b}}$
$\mathbf{S}^*\overline{\mathbf{A}}$	$\mathbf{S}^*$	$\mathbf{S}^*\overline{\mathbf{b}}$

Consider the following linear program. After introducing slacks x_4, x_5 and x_6 and solving with the Simplex method, we obtain the final Simplex tableau, also given below.

$$
\begin{array}{rrl}
\mathit{Maximize} & 2x_1 - 1x_2 + 1x_3 & \\
\text{s.t.} & 3x_1 - 2x_2 + 2x_3 & \leq 15 \\
& -1x_1 + 4x_2 + 1x_3 & \leq 5 \\
& 1x_1 - 1x_2 - 2x_3 & \leq 10 \\
\text{and} & x_1, x_2, x_3 \geq 0. &
\end{array}
$$

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	Right side
Z	0	1	0	0	0.5	0.7	0.1	0	11
X1	1	0	1	0	1	0.4	0.2	0	7
X2	2	0	0	1	0.5	0.1	0.3	0	3
X6	3	0	0	0	-2.5	-0.3	0.1	1	6

For each of the following questions we will always assume that this is the starting point for the proposed changes. *All subquestions carry approximately equal weight.*

a. Suppose we are only going to change the coefficient of x_1 in the objective function. Calculate the allowable increase and decrease in this coefficient (from its current value of 2) such that the previously optimal basis remains optimal. What is the *smallest* value that the objective function will have in this allowable range?

b. Suppose we are going to add a new variable x_7. It will have coefficient 3 in the objective function, 5 in the first constraint, -10 in the second constraint and 4 in the third constraint. Is the previously optimal basis still optimal? If optimality is lost, what will be the *leaving* basic variable if we re-optimize from this point?

c. Suppose we are only going to change the ≤ 15 part of the program. Calculate the allowable decrease and increase of this value such that the previously optimal basis remains feasible.