Exam **Linear Programming, Tuesday 6th June 2017, 14:00 - 17:00**
Bachelor program Knowledge Engineering, year 2
Tapijnkazerne, building *Z*

- **This exam consist of 7 questions, totalling 100 points. Attempt all questions, although feel free to solve them in any order you like. Note that not all questions have the same number of points! Please motivate all of your answers with arguments or computations.**
- **"True optimization is the revolutionary contribution of modern research to decision processes" (George Dantzig)**

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1. [**15 points**] We have 3 shipping containers. One container will be placed on the *left* of the ship, one in the *middle*, and one on the *right*. The 3 containers each have different volumes (measured in m^3) and have different maximum weight limits (measured in kilograms). They are shown in the following table.

Container	Maximum weight limit (kg)	Volume (m^3)
Left	30	50
Middle	70	30
Right	20	25

There are 4 different types of goods, which yield different profits per kilogram and have different volumes per kilogram. There is also a limited amount of each good available. The relevant numbers are shown in the table below.

Good	Amount available (kg)	Volume per kilogram (m^3 per kg)	Profit per kilogram ($ per kg)
1	10	2.5	3
2	90	1.5	1
3	45	7	5
4	30	3	2

We wish to load goods into the containers to maximimize profit, such that no container exceeds its maximum weight limit or its volume, and subject to the limitations on the amount of each good available. Each container is allowed to carry a mixture of different goods. Finally, to ensure that the ship remains balanced in the water, we require that the ratio of the total weight of goods in the *left* container, and the total weight of goods in the *right* container, lies in the range $[0.95, 1.05]$.

Formulate a linear program to solve this problem. Clearly define your variables, objective function and constraints. Note that you do *not* need to solve the linear program you write.

2. [**15 points**] Consider the following linear program:

$$\begin{array}{rl} Maximize & -3x_1 + 4x_2 + 5x_3 \\ \text{s.t.} & -10x_1 + 5x_2 + 10x_3 \leq 30 \\ & 5x_1 + 4x_2 - 3x_3 \leq 21 \\ & 1x_1 - 1x_2 \leq 50 \\ \text{and} & x_1, x_2\, x_3 \geq 0. \end{array}$$

a. Solve the program using the Simplex method, starting from the usual initial solution (i.e. all decision variables are nonbasic). Show each Simplex tableau clearly. When you are finished, state explicitly the optimum value Z^* of the objective function and the values of the decision variables and the slack variables.

b. For the solution you found in part (a), explain which variables are basic and nonbasic, and give an interpretation of this in terms of the functional constraints and non-negativity constraints of the original program.

3. **[15 points]**

a. Consider the following (primal) linear program:

$$\begin{array}{rll} \textit{Maximize} & -1x_1 + 1x_2 + 2x_3 & \\ \text{s.t.} & 1x_1 + 2x_2 - 1x_3 & \leq 20 \\ & -2x_1 + 4x_2 + 2x_3 & \leq 60 \\ & 2x_1 + 3x_2 + 1x_3 & \leq 50 \\ & x_1, x_2, x_3 \geq 0. & \end{array}$$

Let y_i be the dual decision variable corresponding to the ith primal constraint. You are told that there is an optimal basic feasible solution for the *dual* of the above program with the following properties: y_2 and y_3 are greater than zero and the second dual constraint is not tight. Use this information to derive an optimal basic feasible solution for the original *primal* program.

b. Write down a simple example of a (primal) linear program that is infeasible, such that the dual is unbounded.

4. **[15 points]**

a. Use the SOB method to write down the dual of the following linear program.

$$\begin{array}{rll} \textit{Minimize} & 4x_1 + 5x_2 + 3x_3 & \\ \text{s.t.} & 1x_1 + 1x_2 + 2x_3 & \geq 20 \\ & -15x_1 + 6x_2 - 5x_3 & \leq 50 \\ & 1x_1 + 3x_2 + 5x_3 & = 30 \\ \text{and} & x_1 \geq 0 & \\ & x_2 \text{ unconstrained} & \\ & x_3 \leq 0 & \end{array}$$

b. Suppose we have a Maximization linear program where all the decision variables are constrained to be ≥ 0. There are p constraints of the form ≤ 5, q constraints of the form $= 7$ and r constraints of the form ≥ 8. (So there are $p+q+r$ functional constraints in total). We wish to solve this with the big-M method, so we set up the Simplex tableau to obtain the initial, artificial basic feasible solution. What will the Z value (i.e. the objective function value) be of the initial, artificial basic feasible function, expressed as a function of p, q, r and M?

5. **[15 points]** We have seen during the course that it is possible to describe the Simplex tableaux only in terms of values $\mathbf{b}$, $\mathbf{B}$, $\mathbf{c}$, $\mathbf{c}_B$, and $\mathbf{A}$:

$\mathbf{c}_B\mathbf{B}^{-1}\mathbf{A}-\mathbf{c}$	$\mathbf{c}_B\mathbf{B}^{-1}$	$\mathbf{c}_B\mathbf{B}^{-1}\mathbf{b}$
$\mathbf{B}^{-1}\mathbf{A}$	$\mathbf{B}^{-1}$	$\mathbf{B}^{-1}\mathbf{b}$

Consider the following linear program:

$$\begin{array}{rll} \textit{Maximize} & 5x_1 + 9x_2 + 7x_3 & \\ \text{s.t.} & 2x_1 + 1x_2 + 2x_3 & \leq 8 \\ & 1x_1 + 3x_2 + 2x_3 & \leq 10 \\ & 3x_1 + 4x_2 + 2x_3 & \leq 12 \\ & x_1, x_2, x_3 \geq 0. & \end{array}$$

Let x_4, x_5 and x_6 denote the slack variables for the first, second and third constraints, respectively. Suppose that after some number of iterations of the simplex method, you are shown the incomplete simplex tableau below. **You are also told that no basic variable has a value larger than 3.5**.

```
Bas|Eq|              Coefficient of              | Right
Var|No| Z|   X1     X2     X3     X4     X5     X6    |  side
___|__|__|_______________________________________|______
   |  |  |                                       |
 Z | 0| 1|   ?      ?      ?     0.75     ?      ?    |    33.5
 X6| 1| 0|   2.5    ?      ?     0.5      ?      ?    |    ?
 X?| 2| 0|   ?      0      ?     0.75     ?      ?    |    ?
 X?| 3| 0|   ?      ?      ?    -0.5      ?      ?    |    ?
```

Fill in the missing numbers in the table. Explain your reasoning carefully. Do not just pivot mechanically: use analytical/algebraic arguments!

6. **[12 points]** Let U be a set of n elements, $\{u_1, ..., u_n\}$, where each element $u_i \in U$ has an associated non-negative weight $w(u_i)$. For a subset $U' \subseteq U$, we say that the weight of U' is simply the sum of the weights of the elements in U'. We are given a set of m "targets" $\mathcal{T} = \{T_1, ..., T_m\}$, where each target T_i is a **size-3** subset of U. A subset $U' \subseteq U$ is called a *hitting set* for $\mathcal{T}$ if, for every $T_i \in \mathcal{T}$, $T_i \cap U' \neq \emptyset$ i.e. U' intersects with each target at least once. Ideally, we would like to compute a *minimum-weight* hitting set for $\mathcal{T}$. However, this problem is NP-hard, so exact solutions that run in polynomial time are unlikely.

Use your knowledge of Linear Programming to construct a polynomial-time 3-approximation for this problem. That is, an efficient algorithm which returns a htting set U' whose weight is at most 3 times larger than the true minimum-weight hitting set. Be sure to motivate carefully why your algorithm is correct.

7. **[13 points]** Once we have finished executing the Simplex method we have a Simplex tableau corresponding to an optimal basic feasible solution. Suppose we then change the original linear program in some way. Sensitivity analysis helps us understand the new role of the previously optimal basic feasible solution, without applying the Simplex algorithm from the very beginning again. We often commence sensitivity analysis by writing down the *revised* final Simplex tableau, where $\overline{\mathbf{A}}$, $\overline{\mathbf{b}}$, $\overline{\mathbf{c}}$ are the modified versions of $\mathbf{A}$, $\mathbf{b}$, $\mathbf{c}$ respectively:

$\mathbf{y}^*\overline{\mathbf{A}}-\overline{\mathbf{c}}$	$\mathbf{y}^*$	$\mathbf{y}^*\overline{\mathbf{b}}$
$\mathbf{S}^*\overline{\mathbf{A}}$	$\mathbf{S}^*$	$\mathbf{S}^*\overline{\mathbf{b}}$

Consider the following linear program. After introducing slacks x_4, x_5 and x_6 and solving with the Simplex method, we obtain the final Simplex tableau, also given below.

$$\begin{aligned}
Maximize \quad & 5x_1 + 3x_2 + 5x_3 \\
\text{s.t.} \quad & 1x_1 + 7x_2 - 2x_3 \le 10 \\
& 2x_1 + 9x_2 + 3x_3 \le 15 \\
& 4x_1 + 3x_2 + 2x_3 \le 9 \\
\text{and} \quad & x_1, x_2, x_3 \ge 0.
\end{aligned}$$

```
Bas|Eq|              Coefficient of              | Right
Var|No| Z|   X1     X2     X3     X4     X5     X6   |  side
___|__|__|_____________________________________|______
   |  |  |                                     |
 Z | 0| 1|    5    4.5      0      0      0    2.5  |  22.5
X4 | 1| 0|    5     10      0      1      0      1  |    19
X5 | 2| 0|   -4    4.5      0      0      1   -1.5  |   1.5
X3 | 3| 0|    2    1.5      1      0      0    0.5  |   4.5
```

For each of the following questions we will always assume that this is the starting point for the proposed changes, and that this is the "previously optimal basic solution".

a. Suppose we are only going to change the ≤ 9 part of the program. Calculate the allowable decrease and increase of this value such that the previously optimal basic feasible solution remains feasible. Within this allowable range, what is the rate of change of the objective function?

b. Suppose we are only going to change the coefficient of x_3 in the objective function. Calculate the allowable increase and decrease in this coefficient (from its current value of 5) such that the previously optimal basic feasible solution remains optimal.

c. Suppose we are going to add a new variable x_7. It will have coefficient 6 in the objective function, -3 in the first constraint, 4 in the second constraint and 2 in the third constraint. Is the previously optimal basic feasible solution still optimal? **New:** If optimality is lost, what will be the *leaving* basic variable if we re-optimize from this point?