Exam **Linear Programming, Monday 3rd July 2017, 09:00 - 12:00**
Bachelor program Knowledge Engineering, year 2
Tapijnkazerne, building Z

- **This exam consist of 7 questions, totalling 100 points. Attempt all questions, although feel free to solve them in any order you like. Note that not all questions have the same number of points! Please motivate all of your answers with arguments or computations.**
- **'The real problem is that programmers have spent far too much time worrying about efficiency in the wrong places and at the wrong times." (Donald Knuth)**

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1. **[15 points]** Red Dwarf Toasters needs to produce 1000 of their new "Talkie Toaster". There are three ways this toaster can be produced: manually, semi-automatically, and robotically. Manual assembly requires 1 minute of skilled labour, 40 minutes of unskilled labour, and 3 minutes of assembly room time. The corresponding values for semiautomatic assembly are 4, 30, and 2; while those for robotic assembly are 8, 20, and 4. There are 4500 minutes of skilled labour, 36000 minutes of unskilled labour, and 2700 minutes of assembly room time available for this product. The total cost for producing manually is \$7/toaster; semiautomatically is \$8/toaster; and robotically is \$8.50/toaster.

   a. Formulate the problem of producing 1000 toasters at minimum cost meeting the resource requirements. Clearly define your variables, objective function and constraints.

   b. Our union contract states that the amount of skilled labour time used is at least 10% of the total labour (unskilled plus skilled) time used. Update your formulation in (a) to handle this requirement.

   c. Any unused assembly floor time can be rented out at a profit of \$0.50/minute. Update your formulation to include this possibility.

   Note that you do *not* need to solve the linear programs you write, and fractional production of toasters is permitted.

2. **[15 points]** Consider the following linear program:

$$
\begin{aligned}
\textit{Maximize} \quad & -1x_1 + 1x_2 + 2x_3 \\
\text{s.t.} \quad & 1x_1 + 2x_2 - 1x_3 \leq 20 \\
& -2x_1 + 4x_2 + 2x_3 \leq 60 \\
& 3x_1 + 3x_2 + 1x_3 \leq 50 \\
\text{and} \quad & x_1, x_2\, x_3 \geq 0.
\end{aligned}
$$

   a. Solve the program using the Simplex method, starting from the usual initial solution (i.e. all decision variables are nonbasic). Show each Simplex tableau clearly. When you are finished, state explicitly the optimum value Z^* of the objective function and the values of the decision variables and the slack variables.

   b. For the solution you found in part (a), explain which variables are basic and nonbasic, and give an interpretation of this in terms of the functional constraints and non-negativity constraints of the original program.

3. **[15 points]**

a. Consider the following (primal) linear program:

$$\begin{array}{rll} \textit{Maximize} & -1x_1 + 3x_2 + 2x_3 & \\ \text{s.t.} & 4x_1 + 2x_2 - 1x_3 & \leq 30 \\ & -2x_1 + 4x_2 + 2x_3 & \leq 60 \\ & -2x_1 + 3x_2 + 1x_3 & \leq 50 \\ & x_1, x_2, x_3 \geq 0. & \end{array}$$

Let y_i be the dual decision variable corresponding to the ith primal constraint. You are told that there is an optimal basic feasible solution for the *dual* of the above program with $(y_1, y_2, y_3) = (\frac{1}{3}, \frac{7}{6}, 0)$. Use this information to derive an optimal basic feasible solution for the original *primal* program.

b. (This is unconnected to part a). Suppose we have a Maximization linear program with n decision variables, where all the decision variables are constrained to be ≥ 0. There are p constraints of the form ≤ 4, q constraints of the form $= 5$ and r constraints of the form ≥ 6. (So there are $p + q + r$ functional constraints in total). We wish to solve this with the big-M method, so we set up the Simplex tableau to obtain the initial, artificial basic feasible solution. Which variables will be in the basis at this point, and how many variables are in the basis in total? Which variables will be non-basic at this point, and how many variables are non-basic in total?

4. **[15 points]**

a. A linear program can have exactly one of three states: (F)easible with finite optimum, feasible with (U)nbounded optimum, and (I)nfeasible. The Strong Duality Theorem says that for a primal program P and its dual D only certain combinations of these states can occur. Consider now the (primal) linear program below.

$$\begin{array}{rll} \textit{Maximize} \quad Z = & ax_1 + bx_2 & \\ \text{s.t.} & cx_1 + dx_2 & \leq e \\ & fx_1 + gx_2 & \leq h \\ & x_1, x_2 \geq 0. & \end{array}$$

For each of the combinations that the Strong Duality Theorem says can occur, give values of the constants a, b, c, d, e, f, g, h such that the primal and dual programs you obtain are an example of that combination. To be clear: the constants can take any values you like - negative, non-negative, zero - and you can give a different set of constants per combination.

b. Explain briefly how the Weak Duality Theorem can be combined with the Simplex method, and Bland's pivoting rule, to prove the following statement: a primal linear program is feasible with a finite optimum, if and only its dual is.

5. **[15 points]** We have seen during the course that it is possible to describe the Simplex tableaux only in terms of values $\mathbf{b}$, $\mathbf{B}$, $\mathbf{c}$, $\mathbf{c}_B$, and $\mathbf{A}$:

$\mathbf{c}_B\mathbf{B}^{-1}\mathbf{A} - \mathbf{c}$	$\mathbf{c}_B\mathbf{B}^{-1}$	$\mathbf{c}_B\mathbf{B}^{-1}\mathbf{b}$
$\mathbf{B}^{-1}\mathbf{A}$	$\mathbf{B}^{-1}$	$\mathbf{B}^{-1}\mathbf{b}$

Consider the following linear program:

$$
\begin{array}{rl}
\textit{Maximize} & 2x_1 - 6x_2 + 3x_3 \\
\text{s.t.} & 1x_1 + 3x_2 + 2x_3 \ \leq 50 \\
 & -1x_1 - 2x_2 + 1x_3 \ \leq 20 \\
 & 3x_1 + 6x_2 - 3x_3 \ \leq 80 \\
 & x_1, x_2, x_3 \geq 0.
\end{array}
$$

Let x_4, x_5 and x_6 denote the slack variables for the first, second and third constraints, respectively. Suppose that after some number of iterations of the Simplex method, you are shown the incomplete Simplex tableau below.

```
Bas|Eq|                Coefficient of                  | Right
Var|No| Z|   X1     X2     X3     X4     X5     X6    |  side
___|__|__|_____________________________________________|______
   |  |  |                                             |
 Z | 0| 1|   ?      ?      ?      ?   (-1/3)    ?     |  (230/3)
 X?| 1| 0|   ?      ?      ?      ?   (-2/3)    ?     |  (10/3)
 X?| 2| 0|   ?    (1/3)    ?      ?      ?      ?     |    ?
 X?| 3| 0|   ?      ?      ?      ?      3      ?     |   140
```

Fill in the missing numbers in the table. Explain your reasoning carefully. Do not just pivot mechanically: use analytical/algebraic arguments!

6. **[12 points]** Let $G = (V, E)$ be an undirected graph. Recall that a *vertex cover* of G is a subset $V' \subseteq V$ such that, for every edge $e = \{u, v\} \in E$, $V' \cap \{u, v\} \neq \emptyset$. In other words, every edge has at least one endpoint in V'. We are interested in constructing a vertex cover of *minimum* size. The natural Integer Linear Program for this problem has one binary variable for each vertex, and one constraint per edge:

$$
\begin{array}{rll}
\textit{Minimize} & \sum_{v \in V} x_v & \\
\text{s.t.} & x_u + x_v \geq 1 & \text{for each edge } \{u, v\} \in E \\
\text{and} & x_v \in \{0, 1\} & \text{for each vertex } v \in V
\end{array}
$$

Consider the LP-relaxation of the above ILP, obtained by replacing the constraints $x_v \in \{0, 1\}$ by $0 \leq x_v \leq 1$. A feasible solution to the LP-relaxation is said to be *half-integral* if each variable in the solution has a value in the set $\{0, \frac{1}{2}, 1\}$.

Use the line-segment definition of basic feasible solution to prove that *every* basic feasible solution of the LP-relaxation is half-integral. The easiest way of tackling this is to consider a feasible solution that is not half-integral, and to prove that it cannot possibly be a basic feasible solution: focus on variables with values strictly less than and strictly more than $\frac{1}{2}$.

7. **[13 points]** Once we have finished executing the Simplex method we have a Simplex tableau corresponding to an optimal basic feasible solution. Suppose we then change the original linear program in some way. Sensitivity analysis helps us understand the new role of the previously optimal basic feasible solution, without applying the Simplex algorithm from the very beginning again. We often commence sensitivity analysis by writing down the *revised* final Simplex tableau, where $\overline{\mathbf{A}}$, $\overline{\mathbf{b}}$, $\overline{\mathbf{c}}$ are the modified versions of $\mathbf{A}$, $\mathbf{b}$, $\mathbf{c}$ respectively:

$\mathbf{y}^*\overline{\mathbf{A}}-\overline{\mathbf{c}}$	$\mathbf{y}^*$	$\mathbf{y}^*\overline{\mathbf{b}}$
$\mathbf{S}^*\overline{\mathbf{A}}$	$\mathbf{S}^*$	$\mathbf{S}^*\overline{\mathbf{b}}$

Consider the following linear program. After introducing slacks x_4, x_5 and x_6 and solving with the Simplex method, we obtain the final Simplex tableau, also given below.

$$\begin{array}{rll} \textit{Maximize} & 2x_1 - 1x_2 + 1x_3 & \\ \text{s.t.} & 3x_1 - 2x_2 + 2x_3 & \leq 15 \\ & -1x_1 + 4x_2 + 1x_3 & \leq 5 \\ & 1x_1 - 1x_2 - 2x_3 & \leq 10 \\ \text{and} & x_1, x_2, x_3 \geq 0. & \end{array}$$

```
Bas|Eq|              Coefficient of              | Right
Var|No| Z|  X1    X2    X3    X4    X5    X6  |  side
___|__|__|____________________________________|______
   |  |  |                                    |
 Z | 0| 1|   0     0   0.5   0.7   0.1     0  |    11
 X1| 1| 0|   1     0     1   0.4   0.2     0  |     7
 X2| 2| 0|   0     1   0.5   0.1   0.3     0  |     3
 X6| 3| 0|   0     0  -2.5  -0.3   0.1     1  |     6
```

For each of the following questions we will always assume that this is the starting point for the proposed changes, and that this is the "previously optimal basic solution".

a. Suppose we are going to add a new variable x_7. It will have coefficient 3 in the objective function, 5 in the first constraint, -10 in the second constraint and 2 in the third constraint. Is the previously optimal basic feasible solution still optimal?

b. Suppose we are only going to change the coefficient of x_1 in the objective function. Calculate the allowable increase and decrease in this coefficient (from its current value of 2) such that the previously optimal basic feasible solution remains optimal. **New:** What is the *smallest* value that the objective function can have in this allowable range?

c. Suppose we are only going to change the ≤ 15 part of the program. Calculate the allowable decrease and increase of this value such that the previously optimal basic feasible solution remains feasible. Within this allowable range, what is the rate of change of the objective function?