Exam **Linear Programming, resit, Monday 23rd June 2014, 14:00 - 17:00**
Bachelor program Knowledge Engineering, year 2
Turnzaal, GG90-92

- **This exam consist of 6 questions, totalling 100 points. Attempt all questions, although feel free to solve them in any order you like. Note that not all questions have the same number of points!**
- **Please motivate all of your answers with arguments or computations.**

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1. [**15 points**] We are given three shipping containers A, B and C:

   - Container A has a volume of 7000 cubic feet, and can carry up to 12 tonnes of goods.
   - Container B has a volume of 9000 cubic feet, and can carry up to 18 tonnes of goods.
   - Container C has a volume of 5000 cubic feet, and can carry up to 10 tonnes of goods.

   We have four different types of goods available, in varying amounts. Each good has a specific volume (per tonne) and a specific profit (per tonne):

   - There are up to 20 tonnes of good 1 available. Each tonne of this good occupies 500 cubic feet. Each tonne of this good generates 320 euros of profit.
   - There are up to 16 tonnes of good 2 available. Each tonne of this good occupies 700 cubic feet. Each tonne of this good generates 400 euros of profit.
   - There are up to 25 tonnes of good 3 available. Each tonne of this good occupies 600 cubic feet. Each tonne of this good generates 360 euros of profit.
   - There are up to 13 tonnes of good 4 available. Each tonne of this good occupies 400 cubic feet. Each tonne of this good generates 290 euros of profit.

   Any proportion of the available goods can be selected.

   We want to load goods into the containers to maximize profit, such that no container is overloaded (either in terms of volume or weight). There is an extra condition: to keep the ship balanced, each container should use the same fraction of its total maximum weight. So, for example, if Container A contains 6 tonnes of goods, Container B should contain 9 tonnes of goods and Container C should contain 5 tonnes of goods.

   Formulate a linear program to solve this problem. Clearly define your variables, objective function and constraints. Note that you do *not* need to solve the linear program you write.

2. **[15 points]** Consider the following linear program:

$$\begin{array}{rl} \textit{Maximize} & -4x_1 + 3x_2 + 6x_3 \\ \text{s.t.} & -3x_1 + 1x_2 + 3x_3 \leq 30 \\ & -2x_1 + 2x_2 + 3x_3 \leq 50 \\ \text{and} & x_1, x_2, x_3 \geq 0. \end{array}$$

a. Solve the program using the Simplex method, starting from the usual initial solution (i.e. all decision variables are nonbasic). Show each Simplex tableau clearly. When you are finished, state explicitly the optimum value Z^* of the objective function and the values of the decision variables and the slack variables.

b. For the solution you found in part a), explain which variables are basic and nonbasic, and give an interpretation of this in terms of the constraints of the original program.

c. Write down the dual of the linear program.

d. Without solving the dual explicitly, write down an optimal solution for it (and the corresponding Z^* value), including the values of dual surplus variables. State explicitly where you obtained this information.

3. **[15 points]**

a. Use the SOB method to write down the dual of the following linear program.

$$\begin{array}{rl} \textit{Maximize} & 2x_1 + 5x_2 + 3x_3 \\ \text{s.t.} & 1x_1 - 2x_2 + 1x_3 \leq 70 \\ & 1x_1 - 2x_2 + 1x_3 \geq 20 \\ & 2x_1 + 4x_2 + 1x_3 = 50 \\ \text{and} & x_1, x_3 \geq 0 \\ & x_2 \text{ unbounded} \end{array}$$

b. The Simplex method assumes that all decision variables are constrained to be *non-negative*. Explain briefly how we can adapt the Simplex method to also cope with *unbounded* decision variables (such as "x_2 unbounded" in the program shown above).

4. **[15 points]** We have seen during the course that it is possible to describe the Simplex tableaux only in terms of values $\mathbf{b}$, $\mathbf{B}$, $\mathbf{c}$, $\mathbf{c}_B$, and $\mathbf{A}$:

$\mathbf{c}_B\mathbf{B}^{-1}\mathbf{A}-\mathbf{c}$	$\mathbf{c}_B\mathbf{B}^{-1}$	$\mathbf{c}_B\mathbf{B}^{-1}\mathbf{b}$
$\mathbf{B}^{-1}\mathbf{A}$	$\mathbf{B}^{-1}$	$\mathbf{B}^{-1}\mathbf{b}$

Consider the following linear program:

$$\begin{array}{rll} \textit{Maximize} & 5x_1 + 9x_2 + 7x_3 & \\ \text{s.t.} & 1x_1 + 3x_2 + 2x_3 & \leq 10 \\ & 3x_1 + 4x_2 + 2x_3 & \leq 12 \\ & 2x_1 + 1x_2 + 2x_3 & \leq 8 \\ & x_1, x_2, x_3 \geq 0. & \end{array}$$

Let x_4, x_5 and x_6 denote the slack variables for the first, second and third constraints, respectively. Suppose that after some number of iterations of the simplex method, you are shown the incomplete simplex tableau below. You are also told that, at this point, x_2 is the only basic decision variable.

```
Bas|Eq|                Coefficient of                | Right
Var|No| Z|   X1     X2     X3     X4     X5     X6   |  side
___|__|__|___________________________________________|______
   |  |  |                                           |
 Z | 0| 1|   ?      ?      ?      ?      ?       ?   |     ?
X?| 1| 0|   ?      ?      ?      ?    -0.75     ?   |     1
X2| 2| 0|   ?      ?      ?      ?     0.25     ?   |     3
X?| 3| 0|   ?      ?      ?      ?    -0.25     ?   |     ?
```

Fill in the missing numbers in the table. Explain your reasoning carefully.

5. **[15 points]**

   a. Explain in your own words what the *weak duality theorem* and *strong duality theorem* say.

   b. Consider the following (primal) linear program:

$$\begin{array}{rlll} \textit{Maximize} & Z = & 2x_1 - 1x_2 + 1x_3 & \\ \text{s.t.} & & 3x_1 + 1x_2 + 1x_3 & \leq 60 \\ & & 1x_1 - 1x_2 + 2x_3 & \leq 10 \\ & & 1x_1 + 1x_2 - 1x_3 & \leq 20 \\ & & x_1, x_2, x_3 \geq 0. & \end{array}$$

   Let y_i be the dual decision variable corresponding to the ith primal constraint. You are told that there is an optimal basic feasible solution for the *dual* of the above program with the following properties: y_2 and y_3 are greater than zero and the third dual constraint is not tight. Use this information to derive an optimal basic feasible solution for the original *primal* program.

6. **[25 points]** Once we have finished executing the Simplex method we have a Simplex tableau corresponding to an optimal basic feasible solution. Suppose we then change the original linear program in some way. Sensitivity analysis helps us understand the new role of the previously optimal basic feasible solution, without applying the Simplex algorithm from the very beginning again. We often commence sensitivity analysis by writing down the *revised* final Simplex tableau, where $\overline{\mathbf{A}}$, $\overline{\mathbf{b}}$, $\overline{\mathbf{c}}$ are the modified versions of $\mathbf{A}$, $\mathbf{b}$, $\mathbf{c}$ respectively:

$\mathbf{y}^*\overline{\mathbf{A}}-\overline{\mathbf{c}}$	$\mathbf{y}^*$	$\mathbf{y}^*\overline{\mathbf{b}}$
$\mathbf{S}^*\overline{\mathbf{A}}$	$\mathbf{S}^*$	$\mathbf{S}^*\overline{\mathbf{b}}$

Consider the following linear program. After introducing slacks x_4, x_5 and x_6 and solving with the Simplex method, we obtain the final Simplex tableau, also given below.

$$\begin{array}{rll} \textit{Maximize} & 1x_1 - 7x_2 + 3x_3 & \\ \text{s.t.} & 2x_1 + 1x_2 - 1x_3 & \leq 4 \\ & 4x_1 - 3x_2 + 0x_3 & \leq 2 \\ & -3x_1 + 2x_2 + 1x_3 & \leq 3 \\ \text{and} & x_1, x_2, x_3 \geq 0. & \end{array}$$

```
Bas|Eq|              Coefficient of               | Right
Var|No| Z|  X1    X2    X3    X4    X5    X6  |  side
___|__|__|____________________________________|______
   |  |  |                                    |
 Z | 0| 1|   0   5.5     0     0   2.5     3  |    14
X4| 1| 0|   0  2.25     0     1  0.25     1  |   7.5
X1| 2| 0|   1 -0.75     0     0  0.25     0  |   0.5
X3| 3| 0|   0 -0.25     1     0  0.75     1  |   4.5
```

For each of the following questions we will always assume that this is the starting point for the proposed changes, and that this is the "previously optimal basic solution".

a. Suppose we are only going to change the coefficient of x_1 in the objective function. Calculate the allowable increase and decrease in this coefficient (from its current value of 1) such that the previously optimal basic feasible solution remains optimal.

b. Suppose we are only going to change the ≤ 2 part of the program. Calculate the allowable decrease and increase of this value such that the previously optimal basic feasible solution remains feasible. Comment also on the optimality of the previously optimal basic feasible solution in this range.

c. Suppose we are only going to change the coefficient of x_2 in the objective function. Calculate the allowable increase and decrease in this coefficient (from its current value of -7) such that the previously optimal basic feasible solution remains optimal.

d. Suppose we add a constraint $x_1 + 2x_2 + x_3 \leq 3$. Is the previously optimal basic feasible solution still feasible? Is it still optimal? Update the tableau so that it is ready for re-optimization, if optimality and/or feasibility is lost.

e. Suppose we simultaneously change ≤ 4 to ≤ 6 and ≤ 2 to ≤ 4 and ≤ 3 to ≤ 5. Write down the new values of the basic variables for the previously optimal basic feasible solution. Is it still feasible? Is it still optimal? If it has lost feasibility and/or optimality, explain briefly in words how you could re-optimize from this point.

f. Suppose we are going to add a new variable x_7. It will have coefficient 5 in the objective function, 4 in the first constraint, 1 in the second constraint and 0.5 in the third constraint. Is the previously optimal basic feasible solution still optimal? If it has lost optimality, explain briefly in words how the simplex tableau can be updated so that it is ready for re-optimization.