Exam **Linear Programming, Monday 1st June 2015, 09:00 - 12:00**
Bachelor program Knowledge Engineering, year 2
Location: Tapijnkazerne, building $Z$

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- **This exam consist of 6 questions, totalling 100 points. Attempt all questions, although feel free to solve them in any order you like. Note that not all questions have the same number of points! Please motivate all of your answers with arguments or computations.**
- **May the Force be with you!**

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1. [**15 points**] Each day, a factory makes three types of widget: basic, standard and luxury. Each basic, standard, luxury widget generates 5, 7, 12 euros of profit respectively. The widgets produced need three different components: type $A$, type $B$, and type $C$.

   - Basic widgets need 6 components of type $A$, 6 components of type $B$ and 12 components of type $C$.
   - Standard widgets need 4 components of type $A$, 4 components of type $B$ and 18 of type $C$.
   - Luxury widgets need 2 components of type $A$, 9 components of type $B$ and 6 components of type $C$.

   Each day there are 240 components of type $A$ available, 300 of type $B$ and 900 of type $C$.

   We wish to know how many of each type of widget should be produced (per day) to maximize profit.

   a. Formulate a linear program to solve this problem. Clearly define your variables, objective function and constraints.
   b. Extend the program from (a) such that the number of luxury widgets produced, is at most 20% more or less than the number of standard gadgets produced.
   c. Components that are unused at the end of the day are put into storage. Management decides that these storage costs should be incorporated into the model as follows: each unused component *decreases* the daily profit by 0.20 euro per component. (All component types have the same storage cost per component). Extend or update the program from (a) to reflect this.

   Note that you do *not* need to solve the linear programs you write.

2. **[15 points]** Consider the following linear program:

$$\begin{array}{rl} \textit{Maximize} & -4x_1 + 2x_2 + 3x_3 \\ \text{s.t.} & 2x_1 + 1x_2 + 2x_3 \le 40 \\ & -1x_1 + 2x_2 + 4x_3 \le 15 \\ \text{and} & x_1, x_2, x_3 \ge 0. \end{array}$$

a. Solve the program using the Simplex method, starting from the usual initial solution (i.e. all decision variables are nonbasic). Show each Simplex tableau clearly. When you are finished, state explicitly the optimum value $Z^*$ of the objective function and the values of the decision variables and the slack variables.

b. For the solution you found in part a), explain which variables are basic and nonbasic, and give an interpretation of this in terms of the constraints of the original program.

c. Write down the dual of the linear program.

d. Write down the *shadow prices* of the solution you found in part (a), and explain very briefly in your own words how they can be interpreted.

3. **[15 points]** We have seen during the course that it is possible to describe the Simplex tableaux only in terms of values $\mathbf{b}$, $\mathbf{B}$, $\mathbf{c}$, $\mathbf{c}_B$, and $\mathbf{A}$:

| $\mathbf{c}_B\mathbf{B}^{-1}\mathbf{A} - \mathbf{c}$ | $\mathbf{c}_B\mathbf{B}^{-1}$ | $\mathbf{c}_B\mathbf{B}^{-1}\mathbf{b}$ |
|---|---|---|
| $\mathbf{B}^{-1}\mathbf{A}$ | $\mathbf{B}^{-1}$ | $\mathbf{B}^{-1}\mathbf{b}$ |

Consider the following linear program:

$$\begin{array}{rll} \textit{Maximize} & 1x_1 + 2x_2 + 3x_3 & \\ \text{s.t.} & 2x_1 + 2x_2 + 2x_3 & \le 10 \\ & 1x_1 - 3x_2 + 1x_3 & \le 3 \\ & -1x_1 + 2x_2 - 2x_3 & \le 15 \\ & x_1, x_2, x_3 \ge 0. & \end{array}$$

Let $x_4, x_5$ and $x_6$ denote the slack variables for the first, second and third constraints, respectively. Suppose that after some number of iterations of the simplex method, you are shown the incomplete simplex tableau below. You are told that the first and the third constraints are *not* tight (i.e. not satisfied with equality) and that exactly one decision variable is basic.

```
Bas|Eq|               Coefficient of                | Right
Var|No| Z|  X1    X2    X3    X4    X5    X6  |  side
___|__|__|______________________________________|______
   |  |  |                                      |
 Z | 0| 1|  ?     ?     ?     ?     3     ?   |    9
 X?| 1| 0|  ?     ?     ?     ?    -2     ?   |    4
 X?| 2| 0|  ?     ?     ?     ?     1     ?   |    3
 X?| 3| 0|  ?     ?     ?     ?     2     ?   |    ?
```

Fill in the missing numbers in the table. Explain your reasoning carefully.

4. **[15 points]** Consider the following linear program; we refer to this as the *primal* program.

$$\begin{array}{rl} \textit{Maximize} & 5x_1 + 9x_2 + 7x_3 \\ \text{s.t.} & 1x_1 + 3x_2 + 2x_3 \leq 10 \\ & 3x_1 + 4x_2 + 2x_3 \leq 12 \\ \text{and} & x_1, x_2, x_3 \geq 0. \end{array}$$

a. Use weak duality to prove that the optimal value for the primal program cannot exceed 44.

b. Consider the basic feasible solution $B$ for the primal program in which $x_2 > 0$ and the first constraint is not tight. Construct the corresponding complementary basic feasible solution for the *dual* program.

c. Use your answer to (b) to determine whether $B$ is optimal for the primal program. If it is, state the optimum value $Z^*$ it achieves. If it is not, state what the entering basic variable will be if one continues to optimize from this point.

5. **[15 points]** This question concerns - from different perspectives - the *feasibility* of linear programs. The subquestions are unrelated.

a. Consider the following linear program. Construct the complete first Simplex tableau for the first phase of the *two-phase* method and identify the corresponding initial (artificial) basic feasible solution. Also identify the initial entering basic variable and leaving basic variable. You do not need to continue further than this.

$$\begin{array}{rl} \textit{Maximize} & -1x_1 - 2x_2 - 3x_3 \\ \text{s.t.} & 4x_1 + 5x_2 + 6x_3 \geq 7 \\ & 1x_1 + 1x_2 + 1x_3 \geq 3 \\ \text{and} & x_1, x_2, x_3 \geq 0. \end{array}$$

b. Suppose we are given a linear program in its standard form, where (as usual) $A$ is a constraint matrix with $m$ rows and $n$ columns, and $x$ is a column vector of $n$ decision variables:

$$\begin{array}{rl} \textit{Maximize} & cx \\ \text{s.t.} & Ax \leq b \\ \text{and} & x \geq 0. \end{array}$$

Now, suppose we construct the following linear program in $n + m$ decision variables where $x$ is a column vector of $n$ decision variables and $y$ is a row vector of $m$ decision variables. Under which circumstances is this program feasible, and what is the significance of an arbitrary feasible solution to this program?

$$\begin{array}{rl} \textit{Maximize} & \text{whatever objective function you like...} \\ \text{s.t.} & Ax \leq b \\ & yA \geq c \\ & cx = yb \\ \text{and} & x \geq 0, y \geq 0 \end{array}$$

6. **[25 points]** Once we have finished executing the Simplex method we have a Simplex tableau corresponding to an optimal basic feasible solution. Suppose we then change the original linear program in some way. Sensitivity analysis helps us understand the new role of the previously optimal basic feasible solution, without applying the Simplex algorithm from the very beginning again. We often commence sensitivity analysis by writing down the *revised* final Simplex tableau, where $\overline{\mathbf{A}}$, $\overline{\mathbf{b}}$, $\overline{\mathbf{c}}$ are the modified versions of $\mathbf{A}$, $\mathbf{b}$, $\mathbf{c}$ respectively:

| $\mathbf{y}^*\overline{\mathbf{A}}-\overline{\mathbf{c}}$ | $\mathbf{y}^*$ | $\mathbf{y}^*\overline{\mathbf{b}}$ |
|---|---|---|
| $\mathbf{S}^*\overline{\mathbf{A}}$ | $\mathbf{S}^*$ | $\mathbf{S}^*\overline{\mathbf{b}}$ |

Consider the following linear program. After introducing slacks $x_4, x_5$ and $x_6$ and solving with the Simplex method, we obtain the final Simplex tableau, also given below.

$$\begin{array}{rll} \textit{Maximize} & 2x_1 - 1x_2 + 1x_3 & \\ \text{s.t.} & 3x_1 + 1x_2 + 1x_3 & \leq 60 \\ & 1x_1 - 1x_2 + 2x_3 & \leq 10 \\ & 1x_1 + 1x_2 - 1x_3 & \leq 20 \\ \text{and} & x_1, x_2, x_3 \geq 0. & \end{array}$$

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Bas|Eq|                  Coefficient of                | Right
Var|No| Z|   X1     X2     X3     X4     X5     X6   |  side
___|__|__|_____________________________________________|______
   |  |  |                                             |
 Z | 0| 1|    0      0    1.5      0    1.5    0.5   |    25
X4| 1| 0|    0      0      1      1     -1     -2   |    10
X1| 2| 0|    1      0    0.5      0    0.5    0.5   |    15
X2| 3| 0|    0      1   -1.5      0   -0.5    0.5   |     5
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For each of the following questions we will always assume that this is the starting point for the proposed changes, and that this is the "previously optimal basic solution".

a. Suppose we are going to add a new variable $x_7$. It will have coefficient 4 in the objective function, 3 in the first constraint, 2 in the second constraint and 1 in the third constraint. Is the previously optimal basic feasible solution still feasible? Is it still optimal? If it has lost feasibility and/or optimality, explain briefly in words how you could re-optimize from this point.

b. Suppose we simultaneously change $\leq 60$ to $\leq 20$ and $\leq 10$ to $\leq 70$ and $\leq 20$ to $\leq 40$. Write down the new values of the basic variables for the previously optimal basic feasible solution. Is it still feasible? Is it still optimal? If it has lost feasibility and/or optimality, explain briefly in words how you could re-optimize from this point.

c. Suppose we are only going to change the $\leq 10$ part of the program. Calculate the allowable decrease and increase of this value such that the previously optimal basic feasible solution remains feasible. Comment also on the optimality of the previously optimal basic feasible solution in this range.

d. Suppose we add a new constraint: $2x_1 + 3x_2 - 5x_3 \leq 45$. Is the previously optimal basic feasible solution still feasible? Is it still optimal? Update the tableau so that it is ready for re-optimization, if optimality and/or feasibility is lost.

e. Suppose we are only going to change the coefficients of $x_3$ in the objective function. Calculate the allowable increase and decrease in this coefficient (from its current value of 1) such that the previously optimal basic feasible solution remains optimal.

f. Suppose we are only going to change the coefficients of $x_2$ in the objective function. Calculate the allowable increase and decrease in this coefficient (from its current value of -1) such that the previously optimal basic feasible solution remains optimal. **New:** Within this allowable range, how large can the new optimum become?