Maastricht University

Department of Data Science and Knowledge Engineering

Linear Programming 2019/2020
Resit Exam Questions

— Do not turn this page before the official start of the exam! —

First Name, Surname: ______________________________

Student ID: ______________________________

Program: Bachelor Data Science and Knowledge Engineering

Course code: KEN2520

Examiner: Dr. Steven Kelk

Date/time: Monday June 29th$^{\text{th}}$ 2020, 09.00-12.00h (actual exam time: 2 hours)

Format: Closed book exam

Allowed aids: Pens, simple (non-programmable) calculator from the DKE-list of allowed calculators.

Instructions to students:

- The exam consists of 6 questions on 14 pages (excluding the 1 cover page(s)).
- Ensure that you properly motivate your answers.
- Write in a readable way. Answers that cannot be read easily cannot be graded and may therefore lower your grade.
- You are not allowed to have a communication device within your reach, nor to wear or use a watch.
- If you think a question is ambiguous, or even erroneous, and you cannot ask during the exam to clarify this, explain this in detail in your answer to the question.
- If you have not registered for the exam, your answers will not be graded, and thus handled as invalid.
- **Good luck!**

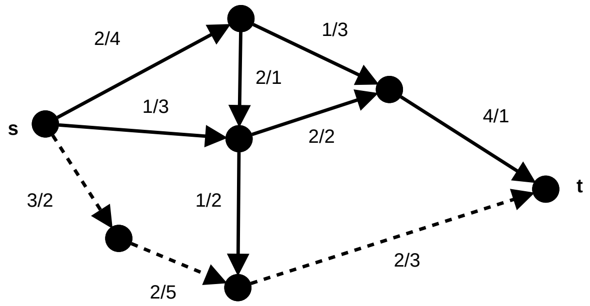
The following table will be filled by the examiners:

Question:	1	2	3	4	5	6	Total
Points:	15	15	15	15	15	15	90
Score:							

Question 1 (15 points)
The minimum-cost flow problem is as follows. We are given a flow network, such as the one shown below, and a certain amount of liquid F. The idea is that F units of liquid are going to be pumped into the system at s and leave the system at t. Each arc of the flow network represents a pipe along which liquid can flow. For every node (except s and t), the amount of liquid that flows into a node must be exactly equal to the amount of liquid that flows out of the node.

Liquid can only flow in the direction of the arc, and each arc has a capacity which gives an upper bound on the amount of liquid that can pass through the arc. Each arc also has a cost which gives the cost per unit of liquid flowing through the arc. In the figure below, the capacity and the cost of an arc is indicated by the numbers "capacity/cost" next to the arc. The minimum-cost flow problem asks this: what is the minimum total cost of sending exactly F units of liquid from s to t, subject to these constraints?

As an example, suppose you send two units of liquid along the dotted path. The cost of this is then $(2 \times 2) + (2 \times 5) + (2 \times 3) = 20$.



Formulate a linear program to solve the minimum-cost flow problem, for a given amount of liquid F. If you wish you can use the figure to illustrate your answer. Clearly define your variables, objective function and constraints. Note that you do not need to solve the linear program you write, and fractional flow in the arcs is permitted.

Question 2 (15 points)
Consider the following linear program:

$$\begin{array}{rl} \textit{Maximize} & 3x_1 + 4x_2 + 5x_3 \\ \text{s.t.} & 2x_1 - 3x_2 + 2x_3 \leq 30 \\ & 4x_1 + 2x_2 + 4x_3 \leq 20 \\ \text{and} & x_1, x_2, x_3 \geq 0. \end{array}$$

a. Solve the program using the Simplex method, starting from the usual initial solution (i.e. all decision variables are nonbasic). Show each Simplex tableau clearly. When you are finished, state explicitly the optimum value Z^* of the objective function and the values of the decision variables and the slack variables.

b. Is the optimal solution you obtained in (a) degenerate? Motivate your answer briefly. One or two sentences is enough.

Question 3 (15 points)
Consider the following linear program; we refer to this as the *primal* program.

$$\begin{aligned} \textit{Maximize} \quad & 2x_1 + 4x_2 + 7x_3 \\ \text{s.t.} \quad & -2x_1 - 3x_2 + 2x_3 \le 30 \\ & 5x_1 + 2x_2 + 5x_3 \le 50 \\ \text{and} \quad & x_1, x_2, x_3 \ge 0. \end{aligned}$$

a. Write down the dual of the program. Then use the weak duality theorem to prove that the optimal value for the primal program cannot exceed 100.

b. Consider the basic feasible solution B for the primal program in which $x_2 > 0$ and $x_4 > 0$ where x_4 is the slack variable for the first functional constraint. Construct the corresponding complementary basic solution for the *dual* program.

c. The coefficient of x_2 in the second functional constraint is currently 2. If this coefficient is reduced to 0 or less, this changes the status of the primal problem. What happens to it? Use the Strong Duality Theorem to motivate your answer.

Question 4 (15 points)

We have seen during the course that it is possible to describe the Simplex tableaux only in terms of values $\mathbf{b}$, $\mathbf{B}$, $\mathbf{c}$, $\mathbf{c}_B$, and $\mathbf{A}$:

$\mathbf{c}_B\mathbf{B}^{-1}\mathbf{A}-\mathbf{c}$	$\mathbf{c}_B\mathbf{B}^{-1}$	$\mathbf{c}_B\mathbf{B}^{-1}\mathbf{b}$
$\mathbf{B}^{-1}\mathbf{A}$	$\mathbf{B}^{-1}$	$\mathbf{B}^{-1}\mathbf{b}$

Consider the following linear program:

$$\begin{array}{rll} \mathit{Maximize} & 1x_1 + 2x_2 + 3x_3 & \\ \text{s.t.} & -5x_1 + 5x_2 + 5x_3 & \leq 10 \\ & 8x_1 + 9x_2 + 15x_3 & \leq 45 \\ & 12x_1 + 13x_2 + 1x_3 & \leq 5 \\ & x_1, x_2, x_3 \geq 0. & \end{array}$$

Let x_4, x_5 and x_6 denote the slack variables for the first, second and third constraints, respectively. Suppose that after some number of iterations of the simplex method, you are shown the incomplete simplex tableau below. You are also told that, in the complementary dual solution (corresponding to this tableau) the dual decision variables (y_1, y_2, y_3) have value $(0.6, 0, 0)$.

```
Bas|Eq|              Coefficient of              | Right
Var|No| Z|   X1     X2     X3     X4     X5     X6  |  side
___|__|__|_____________________________________|______
   |  |  |                                     |
 Z | 0| 1|   ?      ?      ?      ?      ?      ?   |     ?
X3| 1| 0|   ?      ?      ?     0.2     ?      ?   |     2
X?| 2| 0|   ?      ?      ?      -3     ?      ?   |    15
X?| 3| 0|   ?      ?      ?    -0.2     ?      ?   |     ?
```

Fill in the missing numbers in the table. Explain your reasoning carefully. Do not just pivot mechanically: use analytical/algebraic arguments to justify your answers!

Question 5 (15 points)

Let $G = (V, E)$ be an undirected graph. Recall that a *matching* of G is a subset $E' \subseteq E$ such that, for every $v \in V$, v is incident to (i.e. is an endpoint of) at most one edge in E'. We are interested in constructing a matching of *maximum* size. In this variant there are no weights on the edges.

The natural Integer Linear Program (ILP) for this problem has one binary variable x_e for each edge $e \in E$ and one functional constraint per vertex.

a. Write down the natural ILP for this problem, and its LP-relaxation.

Suppose that G is *bipartite*. That is, V can be partitioned into two non-empty, non-overlapping subsets V_L, V_R such that each edge in E has one endpoint in V_L and one endpoint in V_R.

Now, consider a feasible solution x to the LP-relaxation described in (a). Let $E' \subseteq E$ be those edges whose decision variables are fractional in x. Assume $E' \neq \emptyset$.

b. Suppose that E' contains a cycle C. Prove that, by appropriately perturbing the values of the edges in C, two new feasible solutions x', x'' can be obtained such that $x = \frac{1}{2}(x' + x'')$ and x, x', x'' are all distinct.

c. Suppose that E' does *not* contain a cycle C. In this case, consider a longest path P that is contained in E'. We wish to find x', x'' with the same properties as described in (b). How can the argument you gave in (b) be adapted for this purpose?

Question 6 (15 points)

Once we have finished executing the Simplex method we have a Simplex tableau corresponding to an optimal basic feasible solution. Suppose we then change the original linear program in some way. Sensitivity analysis helps us understand the new role of the previously optimal basis, without applying the Simplex algorithm from the very beginning again. We often commence sensitivity analysis by writing down the *revised* final Simplex tableau, where $\overline{\mathbf{A}}$, $\overline{\mathbf{b}}$, $\overline{\mathbf{c}}$ are the modified versions of $\mathbf{A}$, $\mathbf{b}$, $\mathbf{c}$ respectively:

$\mathbf{y}^*\overline{\mathbf{A}}-\overline{\mathbf{c}}$	$\mathbf{y}^*$	$\mathbf{y}^*\overline{\mathbf{b}}$
$\mathbf{S}^*\overline{\mathbf{A}}$	$\mathbf{S}^*$	$\mathbf{S}^*\overline{\mathbf{b}}$

Consider the following linear program. After introducing slacks x_4, x_5 and x_6 and solving with the Simplex method, we obtain the final Simplex tableau, also given below.

$$\begin{array}{rll} \mathit{Maximize} & -3x_1 + 7x_2 + 10x_3 & \\ \text{s.t.} & 1x_1 + 7x_2 - 2x_3 & \leq 20 \\ & -2x_1 - 9x_2 + 3x_3 & \leq 17 \\ & 4x_1 + 3x_2 + 2x_3 & \leq 9 \\ \text{and} & x_1, x_2, x_3 \geq 0. & \end{array}$$

```
Bas|Eq|                Coefficient of               | Right
 Var|No| Z|  X1     X2     X3     X4     X5     X6  |  side
 ___|__|__|_________________________________________|______
    |  |  |                                         |
  Z | 0| 1|  23      8      0      0      0      5  |    45
  X4| 1| 0|   5     10      0      1      0      1  |    29
  X5| 2| 0|  -8  -13.5      0      0      1   -1.5  |   3.5
  X3| 3| 0|   2    1.5      1      0      0    0.5  |   4.5
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For each of the following questions we will always assume that this is the starting point for the proposed changes. *All subquestions carry approximately equal weight.*

a. Suppose we are only going to change the coefficient of x_3 in the objective function. Calculate the allowable increase and decrease in this coefficient (from its current value of 10) such that the previously optimal basic feasible solution remains optimal.

b. Suppose we are only going to change the ≤ 9 part of the program. Calculate the allowable decrease and increase of this value such that the previously optimal basic feasible solution remains feasible. Within this allowable range, what is the rate of change of the objective function, for each unit increase/decrease in the "9"?

c. Suppose we add a constraint $1x_1 + 2x_2 + 4x_3 \leq 17$. Is the previously optimal basic feasible solution still feasible? Suppose you update the simplex tableau to incorporate this new constraint. What will the new set of basic variables be, and what values will they have? (You don't need to actually update the tableau, unless it helps).