6️⃣

Sinusoidal fidelity

Status	Done
Files	
Link	
Type	Lecture
Course	Mathematical Modelling

❗ Exs 23 for the phase shift angle (for example $-\pi/2$), it does not matter whether it is negative or positive, as long as they are equivalent

Sinusoidal Fidelity

- Zafuq is that? Basically how a stable system reacts to sinusoidal input
- Examples of what this input could be

$$\text{input} \quad u(t) = \begin{cases} \sin(\omega t) \\ \cos(\omega t) \\ e^{i\omega t} = \cos(\omega t) + i\sin(\omega t) \end{cases}$$

- For stable systems, any sinusoidal input signal with a given frequency will produce a sinusoidal output with the same frequency, but with a different amplitude (amplified or damped) and subject to a phase shift
- In the lecture notes and wolf vision notes he shows why that is by doing partial fraction expansion and showing how all terms go to 0 except the one with the input but it has a coefficient (thats the scaling/damping part)
- Which means that we can compute the steady state output (~~not sure what steady state means exactly, but does not matter its what we always calculate in the exercises,~~ if you forgot, like me check my notes 1, but its basically x[k+1] = x[k]) from the transfer function, yay!!
- Since the frequency response will determine what will happen with the input signal at frequency ω, then we can do some fancy stuff. Frequencies near the zeros of the transfer function will be attenuated. Frequencies near the poles of the transfer function will be amplified. If poles and zeros are in the same place, they will cancel each other.

> ❗ TLDR: you need to be able to apply one of these formulas depending on the input and whether it is a continuous/discrete system (they have one of them at the bottom of table 7 in the formula sheet, the cos and sin are kind of interchangeable) and the whole atan2 thing is important so you get the phase shift correct, its table 4 in formula sheet, to see how to use them check the exercises

$$y_s(t) = |H(i\omega)| \sin(\omega t + \angle H(i\omega)). \quad y_s(t) = |H(i\omega)| \cos(\omega t + \angle H(i\omega)).$$

$$y_s[k] = |H(e^{i\Omega})| \sin(\Omega k + \angle H(e^{i\Omega})). \quad y_s[k] = |H(e^{i\Omega})| \cos(\Omega k + \angle H(e^{i\Omega})).$$

$$atan2(y,x) = \begin{cases} \arctan\left(\frac{y}{x}\right) & \text{if x} > 0 \\ \arctan\left(\frac{y}{x}\right) + \pi & \text{if x} < 0 \text{ and y} \geq 0 \\ \arctan\left(\frac{y}{x}\right) - \pi & \text{if x} < 0 \text{ and y} < 0 \\ \frac{\pi}{2} & \text{if x} = 0 \text{ and y} > 0 \\ -\frac{\pi}{2} & \text{if x} = 0 \text{ and y} < 0 \\ undefined & \text{if x} = 0 \text{ and y} = 0 \end{cases}$$

Exercises

22

Ex. 22 ①

$$y_s(t) = |H(i\omega)|\, e^{i(\omega t + \angle H\, i\omega)} \qquad u(t) = 5\cos(3t)$$
$$\omega = 3$$

$$y_s(t) = 5\,|H(3i)|\, \cos\left(3t + \angle H(3i)\right)$$

$$H(3i) = \frac{(3i)^2 - 3(3i) + 2}{(3i)^2 + 4(3i) + 3} = \frac{9i^2 - 9i + 2}{9i^2 + 12i + 3}$$

$$= \frac{-9i - 7}{12i - 6} \cdot \frac{(-6 - 12i)}{(-6 - 12i)}$$

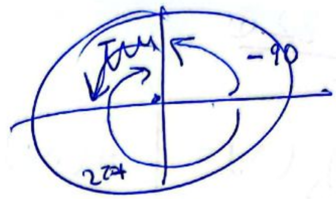
$$= \frac{-11 + 23i}{30}$$

$$|H(3i)| = \sqrt{\left(\frac{-11}{30}\right)^2 + \left(\frac{23}{30}\right)^2} \approx 0.85$$

$$\angle H(3i) = \operatorname{atan2}\left(\frac{23}{\cancel{30}}, \frac{-11}{\cancel{30}}\right) = -1.125 + \pi = 2.0169$$

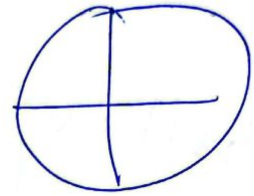
$$y_s(t) = 5(0.85)\cos(3t + 2.0169)$$
$$= 4.25\cos(3t + 2.0169)$$

Ex 22 (2)



again

$$\frac{s^2-3s+2}{s^2+4s+3} \qquad u(t) = \underbrace{5}_{\text{Scalar.}}\cos(\underbrace{3}_{\omega}t)$$



$$y_s(t) = \left| G(i\omega) \right| \sin\left(\omega t + \angle G(i\omega)\right)$$

2π -

$$G(3i) = \frac{(3i)^2 - 3(3i) + 2}{(3i)^2 + 4(3i) + 3}$$

$$= \frac{9i^2 - 9i + 2}{9i^2 + 12i + 3} = \frac{-9i - 7}{12i - 6} \times \frac{-1-2i}{-1-2i}$$

$$= \frac{9i + 7 + 18i^2 + 14i}{-12i + 6 - 24i^2 + 12i} = \frac{23i - 11}{30}$$

$$\left| G(3i) \right| = \sqrt{\left(\frac{11}{30}\right)^2 + \left(\frac{23}{30}\right)^2} = \frac{\sqrt{26}}{6}$$

$$\angle G(3i) = \operatorname{atan2}\left(\frac{23}{30}, -\frac{11}{30}\right) = 2.0169$$

$$y_s(t) = 4.249 \sin\left(3t + 2.0169\right)$$

23

Ex. 23 ①

Q1) $\dfrac{s^2+2s+3}{s^3+4s^2+2s+3}$ $\quad u(t)=\cos(t) \quad \alpha = 1 \quad \omega = 1$

$$y_s(t) = |G(i\omega)| \cos(\omega t + \angle G(i\omega))$$

$$G(i) = \frac{i^2+2i+3}{i^3+4i^2+2i+3} = \frac{2i+2}{i-1} \cdot \frac{-1-i}{-1-i}$$

$$= \frac{-2i-2-2i^2-2i}{-i+1-i^2+i} = \frac{0-4i}{2} = -2i$$

$$|G(i)| = \sqrt{2^2} = 2$$

$$\angle G(i) = \operatorname{atan2}(y,x) = \operatorname{atan2}(-2,0) = -\frac{\pi}{2}$$

$$y_s(t) = \cancel{2\cos\left(t+\frac{3}{4}\pi\right)}$$

$$= 2\cos\left(t - \frac{\pi}{2}\right)$$

Ex 23 ②

Q2) $\frac{9s^2+6s+2}{s^3+5s^2+7s+3}$ $u(t) = 3\sin(2t)$

$\alpha = 3$ $\omega = 2$.

$y_s(t) = |G(i\omega)| \sin(\omega t + \angle G(i\omega))$

$$G(2i) = \frac{9(2i)^2+6(2i)+2}{(2i)^3+5(2i)^2+7(2i)+3} = \frac{\overset{36}{\cancel{34}}i^2+12i+2}{8i^3+20i^2+14i+3}$$

$$= \frac{-\overset{34}{\cancel{30}}+12i}{-17+6i} \times \frac{-17-6i}{-17-6i} =$$

$$= \frac{578 - \cancel{204i} + \cancel{204i} + 72\cancel{2}}{289-36i^2} = \frac{650}{32\cancel{5}} = 2$$

$x = 2$ $y = 0$.

$|G(2i)| = \sqrt{(2)^2} = 2$

$\angle G(2i) = \text{arctan2}(0,2) = 0$ no phase shift.

$y_s(t) = \overset{6}{\cancel{3\times 2}} \sin(2t)$

Ex 23 ③

$u(t) = \sin(t)$

Q3) $\dfrac{s^2 - 3s + 2}{s^3 + 5s^2 + 7s + 3}$ $\quad \alpha = 1 \quad \omega = 1$

$y_s(t) = \alpha\, |G(i\omega)| \sin(\omega t + \angle G(i\omega))$

$$G(i) = \frac{i^2 - 3i + 2}{-i + 5i^2 + 7i + 3} = \frac{+1 - 3i}{-2 + 6i} \times \frac{(-1-3i)}{(-1-3i)}$$

$$= \frac{-1 + \cancel{3i - 3i} + 9i^2}{2 - \cancel{6i + 6i} - 18i^2} = \frac{-10}{20} = -\frac{1}{2}$$

$x = -\frac{1}{2} \quad y = 0.$

$|G(i)| = \frac{1}{2}$

$\angle Gi = \text{artan}\, 2\left(0, -\frac{1}{2}\right) = \pi$

$y_s(t) = \frac{1}{2} \sin(t + \pi)$

Ex 23 ⊙

$u(t) = 3\cos(2t)$

Q4) $\dfrac{-s^2+4s-12}{s^2+4s+12}$ $\alpha = 3$ $\omega = 2$

$y_s'(t) = \alpha |G(i\omega)| \cos(\omega t + \angle G(i\omega))$

$G(2i) = \dfrac{-(2i)^2 + 4(2i) - 12}{(2i)^2 + 4(2i) + 12} = \dfrac{-4i^2 + 8i - 12}{4i^2 + 8i + 12}$

$= \dfrac{-8 + 8i}{8 + 8i} = \dfrac{-1 + i}{1 + i} \cdot \dfrac{1 - i}{(1 - i)}$

$= \dfrac{-1 + i + i - i^2}{2} = \dfrac{2i}{2} = i$

$x = 0 \quad y = 1$

$|G(2i)| = 1$

$\angle G(2i) = \arctan(1, 0) = \frac{\pi}{2}$

$y_s(t) = 3\cos\left(2t + \frac{\pi}{2}\right)$

Ex23 ⑤

Q5)
$$H(s) = \frac{-s^2+4s-5}{s^2+4s+5} \qquad u(t) = \sin(5t)$$
$$\alpha = 1 \qquad \omega = 5$$

$$y_s(t) = \alpha \left| G(i\omega) \right| \sin\left(\omega t + \angle G(i\omega)\right)$$

$$G(5i) = \frac{-(5i)^2 + 4(5i) - 5}{(5i)^2 + 4(5i) + 5} = \frac{-25i^2 + 20i - 5}{25i^2 + 20i + 5}$$

$$= \frac{20 + 20i}{-20 + 20i} = \frac{1 + i}{-1 + i} \cdot \frac{(-1-i)}{(-1-i)} =$$

$$= \frac{-1 - i - i - i^2}{1 - \cancel{i} + \cancel{i} - i^2} = \frac{-2i}{2} = -i$$

$$x = 0 \quad y = -1$$

$$\left| G(5i) \right| = 1$$

$$\angle G(5i) = \operatorname{artan2}(-1, 0) = \frac{-\pi}{2}$$

$$y_s(t) = \sin\left(5t - \frac{\pi}{2}\right) = \sin\left(5t + \frac{3}{2}\pi\right)$$

Ex 23 6

6) $H(s) = \dfrac{s^2 + 2s + 8}{s^3 + 3s^2 + 4s + 8}$ $\quad u(t) = \sin(2t)$

$\omega = 2 \quad \alpha = 1$

$$y_s(t) = \alpha \, |G(i\omega)| \sin\left(\omega t + \angle G(i\omega)\right)$$

$$G(2i) = \frac{(2i)^2 + 2(2i) + 8}{(2i)^3 + 3(2i)^2 + 4(2i) + 8}$$

$$= \frac{4i^2 + 4i + 8}{-8i^3 + 12i^2 + 8i + 8} = \frac{4 + 4i}{-4} = -1 - i$$

$x = -1 \quad y = -1$

$$|G(2i)| = \sqrt{2}$$

$$\angle G(2i) = \arctan(-1, -1) = -\frac{3}{4}\pi$$

$$y_s(t) = \sqrt{2} \sin\left(2t - \frac{3}{4}\pi\right)$$

$$= \sqrt{2} \sin\left(2t + \frac{5}{4}\pi\right)$$

Ex 23 (2)

$u(t) = 5\cos(2t)$

Q7) $H(s) = \dfrac{s^2+16}{s^3+3s^2+4s+2}$

$\omega = 2 \quad \alpha = 5$

$y_s(t) = \alpha \, |G(i\omega)| \cos(\omega t + \angle G(i\omega))$

$$G(2i) = \frac{(2i)^2+16}{(2i)^3 + 3(2i)^2 + 4(2i) + 2} =$$

$$= \frac{12}{-8i - 12 + 8i + 2} = \frac{12}{-10} = -\frac{6}{5}$$

$x = -\frac{6}{5} \quad y = 0$

$$|G(2i)| = \frac{6}{5}$$

$$\angle G(2i) = \operatorname{atan2}\left(0, -\frac{6}{5}\right) = \pi$$

$$y_s(t) = 6 \cos(2t + \pi)$$

Ex23 Q8 removed from notes.

Q8 $H(s) = \dfrac{s^2+5s+9}{s^3+8s^2+9s+2}$ $\qquad u(t) = 5\sin(3t)$

$\alpha = 5 \qquad \omega = 3$

$$y_s(t) = \alpha \left| G(i\omega) \right| \sin\left(\omega t + \angle G(i\omega)\right)$$

$$G(3i) = \frac{\cancel{(3i)^2} + 5(3i) \cancel{+9}}{(3i)^3 + 8(3i)^2 + 9(3i) + 2} = -\frac{15i}{70}$$

$$\cancel{-27i} - 72 + \cancel{27i} + 2$$

$$= -\frac{3}{14}i \qquad x = 0 \qquad y = -\frac{3}{14}$$

$$\left| G(3i) \right| = \frac{3}{14}$$

$$\angle G(3i) = \arctan2\left(-\frac{3}{14}, 0\right) = -\frac{\pi}{2}$$

$$y_s(t) = 5 \cdot \frac{3}{14} \sin\left(3t - \frac{\pi}{2}\right)$$

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Ex 24

$$H(z) = \frac{z^2 - \frac{1}{2}z + \frac{1}{6}}{z^2 - \frac{1}{4}z - \frac{1}{8}}$$

$u[k] = 3\sin\left[\frac{2\pi k}{4}\right]$

$\alpha = 3$ $\quad \omega = \frac{2\pi}{4} = \frac{\pi}{2}$

$y_s[k] = |G(e^{i\omega T})| \sin(\omega kT + \angle G(e^{i\omega T}))$ $\quad T = \frac{1}{f}$

$e^{i\frac{\pi}{2}} = e^{i\frac{\pi}{2}} = \cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right) = i$

$$H(i) = \frac{i^2 - \frac{1}{2}i + \frac{1}{6}}{i^2 - \frac{1}{4}i - \frac{1}{8}} = \frac{-\frac{5}{6} - \frac{1}{2}i}{-\frac{9}{8} - \frac{1}{4}i} \times \frac{-\frac{9}{8} + \frac{1}{4}i}{-\frac{9}{8} + \frac{1}{4}i}$$

$= \frac{4}{5} + \frac{4i}{15}$ $\quad x = \frac{4}{5}$ $\quad y = \frac{4}{15}$

$|H(i)| = \frac{4\sqrt{10}}{15}$ $\quad \angle H(i) = \mathrm{atan2}\left(\frac{4}{15}, \frac{4}{5}\right) = 0.32175$

$$y_s[k] = \frac{4\sqrt{10}}{5}\sin\left(\frac{\pi}{2}k + 0.32175\right)$$

Ex 25

$$u[k] = \frac{2}{3}\cos(k\pi)$$

Q1) $\dfrac{6z^2 - 7z + 5}{8z^3 + 4z^2 - 2z - 1}$

$\alpha = \frac{2}{3}$ $\omega = \pi$

$$y_s[k] = \alpha \, |G(e^{i\omega T})| \cos(\omega kT + \angle G(e^{i\omega T}))$$

$$e^{i\pi} = \cos\pi + i\sin\pi = -1$$

$$G(-1) = -6 \qquad x = -6 \; y = 0$$

$$|G(-1)| = 6.$$

$$\angle G(-1) = \text{artan}2(0, -6) = \pi$$

$$y_s[k] = \frac{2}{3} \times 6 \cos(\pi k + \pi)$$

$$= 4\cos(k\pi + \pi)$$

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Ex.25

Q2) $H(z) = \dfrac{\frac{1}{2}z^2 - \frac{1}{2}z + 1}{z^2 - \frac{1}{2}z + \frac{1}{2}}$ $\quad u[k] = \cos(k\pi/2)$

$\alpha = 1 \quad \omega = \frac{\pi}{2}$

$$y_s[k] = \alpha \left| G\left(e^{i\omega T}\right) \right| \cos\left(\omega k T + \angle G\left(e^{i\omega T}\right)\right)$$

$$e^{\frac{i\omega}{2}} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i$$

$$G(i) = \frac{-\frac{1}{2} - \frac{1}{2}i + 1}{-1 - \frac{1}{2}i + \frac{1}{2}} = \frac{\frac{1}{2} - \frac{1}{2}i}{-\frac{1}{2} - \frac{1}{2}i} = \frac{1 - i}{-1 - i} \cdot \frac{-1 + i}{-1 + i}$$

$$= \frac{-1 + i + i + 1}{1 + 1} = \frac{2i}{2} = i$$

$x = 0 \quad y = 1$

$$|G(i)| = 1$$

$$\angle G(i) = \operatorname{arctan2}(1, 0) = \frac{\pi}{2}$$

$$y_s[k] = 1 \cdot 1 \cdot \cos\left(\frac{\pi}{2}k + \frac{\pi}{2}\right)$$

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