# Interconnected systems

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| Status | Done |
| Files | |
| Link | |
| Type | Lecture |
| Course | Mathematical Modelling |



## Interconnected Systems

- What happens when we put more than one system together
- We do not care about what their individual transfer functions is, but more general what happens and we use their transfer function as a block like F(s) or G(s) but also you can just write F or G and suppress the variable as we never substitute anything there in this context

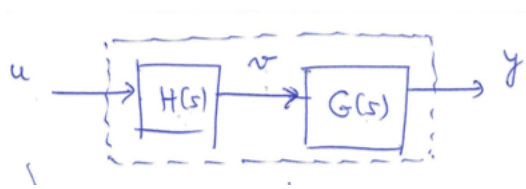
- There is no difference between continuous and linear time systems when it comes to calculating the interconnected system, we only switch to z instead of s in notation, but that even is not necessary if you are suppressing it
- The approach of how to find the affect of each of the connection types on stability and poles is very useful for the type of T/F questions that might come in the exam (he does that in the wolfvision notes), its basically put the first system as k/l and the second as p/q add/multiple based on the connection and see the resulting order and stability properties

# Inverse System

- An *inverse system* is defined to undo the effect of a given system. This means that its transfer function is the reciprocal of the original transfer function: the numerator and denominator polynomials are swapped
- Inverse system causality: if the degree of the denominator and numerator is the same then both the system and its inverse are causal. Number of degrees is important when it comes to causality because it signifies the number of inputs/outputs to the system depending on whether its in the numerator/denominator, if there are more outputs than inputs then the system is not causal because some inputs are in the future (D=0 in the state space if the degree of the numerator and denominator are equal, you can see this in ex 17 when we do fadeev algorithm)
- $H(s) \times H(s)^{-1} = 1$

# Connection Types

## Series (Cascade)




$$Y(s) = G(s)H(s)U(s)$$

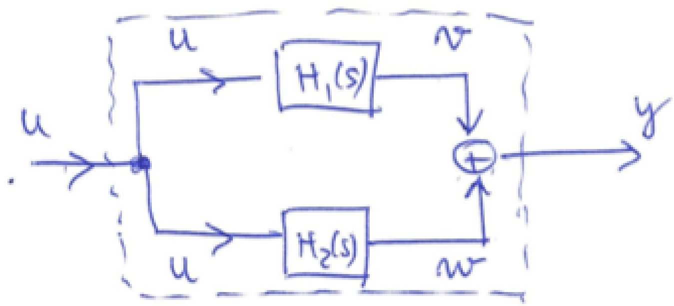
$$\frac{k}{l} \cdot \frac{p}{q} = \frac{p \cdot k}{q \cdot l}$$

$$Order <= n_1 + n_2$$

- Because if we look at the denominator of the combined transfer functions we have q times l, but it might be less if there are repeated poles
- Interestingly, order does not matter, as they are linear systems

$$H_1(s)H_2(s) = H_2(s)H_1(s)$$

## Parallel




❗ When there is a branching, information is copied not split! So the signal at each system is U(s) not 0.5 U(s)

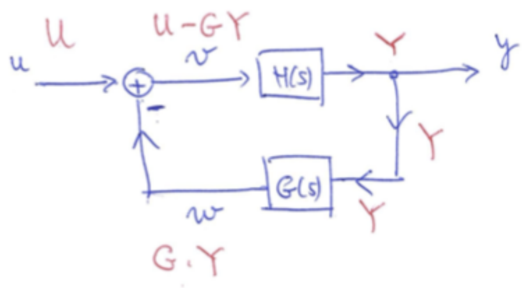
$$Y(s) = (H_1 + H_2)U(s)$$

$$H_1 + H_2 = \frac{p}{q} + \frac{k}{l} = \frac{pl + qk}{ql}$$

$$Order <= n_1 + n_2$$

- Because we also have multiplication in the denominator, similar to the cascade case, also here the order may go down if there are pole zero cancellations

## Feedback




$$Y(s) = \frac{H}{1+HG}U(s)$$

$$\frac{H}{1+HG} = \frac{p/q}{1+\frac{p}{q}\cdot\frac{k}{\ell}}$$
$$= \frac{p\,\ell}{q\cdot\ell + p\cdot k}$$

$$Order <= n_1 + n_2$$

- Order reasoning same as before
- How do we get $Y(s) = \frac{H}{1+HG}U(s)$?
  Good question, whenever there are loops its more tricky than just simple substitution, so you will need to loop forever if you keep substituting, you need to do some tricks to get rid of the loop. The general idea is to introduce variables to the branches (they are given in the exercises usually, also in the exam, but double check that with him as he changes it from year to year) then keep solving the system of equations till you get something with Y on one side and then the only the input U and systems on the other side, so eliminate all the variables you introduced.
  What I do at the beginning is **try to substitute in my head the equations just imagine what the result is and keep substituting, usually fairly quickly you see the problematic variable that keeps coming back in a loop, lets call it e, then focus on e and try to make an equation with e on LHS only on its own and e**

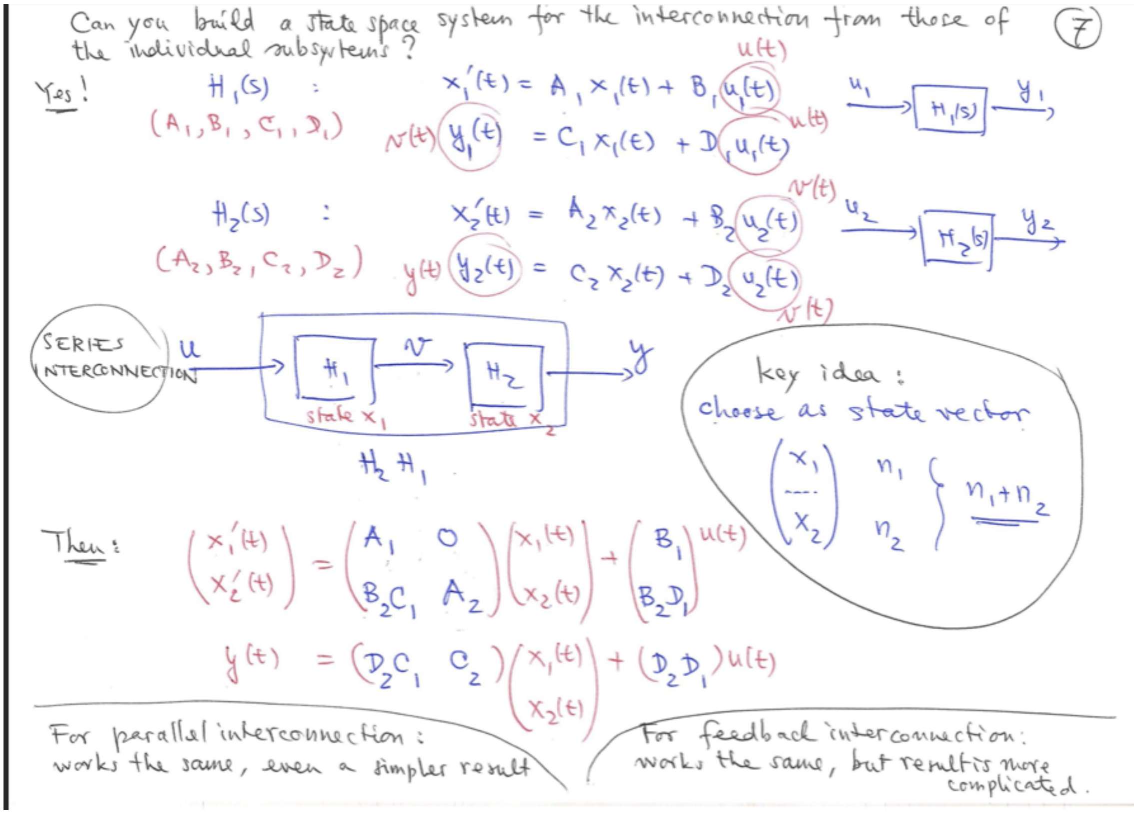
**not appearing on the RHS, usually using the e equation then substituting all the other variables and then factoring or subtracting e out such that you you have what that variable is equal, then plug in that value whenever that variable comes up so it does not come back, usually everything will runs serially from here and its normal substitution. I did my best..... check Ex 21 Q2 for an example**

**You can also see the loop in the diagram so its probably one of those variables in the loop that is problematic**

> ❗ This previous point is relevant if you dont know how to do the interconnected systems questions.

# State Space of Two systems

- This it not in the notes but its really interesting how it works the state space representation of two systems in cascade, its also not likely to come up in an exam
- The state vector is usually the state vectors from each of the two systems in combined in one vector
- The equations for this are given in the formula sheet table 10
- Here is the wolfvision slide:



Can you build a state space system for the interconnection from those of the individual subsystems? (7)

Yes!

$H_1(s)$ : $(A_1, B_1, C_1, D_1)$

$$x_1'(t) = A_1 x_1(t) + B_1 \underbrace{u_1(t)}_{u(t)}$$

$$\underbrace{y_1(t)}_{v(t)} = C_1 x_1(t) + D_1 \underbrace{u_1(t)}_{u(t)}$$

$u_1 \to \boxed{H_1(s)} \to y_1$

$H_2(s)$ : $(A_2, B_2, C_2, D_2)$

$$x_2'(t) = A_2 x_2(t) + B_2 \underbrace{u_2(t)}_{v(t)}$$

$$\underbrace{y_2(t)}_{y(t)} = C_2 x_2(t) + D_2 \underbrace{u_2(t)}_{v(t)}$$

$u_2 \to \boxed{H_2(s)} \to y_2$

SERIES INTERCONNECTION

$u \to \boxed{H_1} \xrightarrow{v} \boxed{H_2} \to y$

state $x_1$ state $x_2$

$H_2 H_1$

key idea: choose as state vector

$$\begin{pmatrix} x_1 \\ \hdashline x_2 \end{pmatrix} \begin{matrix} n_1 \\ n_2 \end{matrix} \Big\} \underline{\underline{n_1 + n_2}}$$

Then:

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \begin{pmatrix} A_1 & 0 \\ B_2 C_1 & A_2 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} B_1 \\ B_2 D_1 \end{pmatrix} u(t)$$

$$y(t) = \begin{pmatrix} D_2 C_1 & C_2 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + (D_2 D_1) u(t)$$

For parallel interconnection: works the same, even a simpler result

For feedback interconnection: works the same, but result is more complicated.

# Stability

| | Subsystems Stable | Subsystems Unstable |
|---|---|---|
| Series or Parallel | Stable | Either |
| Feedback | Either | Either |

# Exercises

## True/False questions on modules

## 20

Exercises: TRUE or FALSE?
If the statement is true, give a brief explanation; if the statement is false, give a counter argument or counter-example.

1. The feedback interconnection of two continuous-time stable systems always produces an overall system which is also stable.
2. If two unstable systems are interconnected by means of a feedback connection, then the resulting overall system is also unstable.
3. If a discrete-time system is unstable, then its inverse system is stable.
4. The overall system resulting from the parallel interconnection of two different systems of order 2 cannot also be of order 2.
5. If a system of order 2 is coupled to another system of order 2 by feedback interconnection, then the overall system can be of order 2.
6. If three transfer functions of orders n1, n2 and n3, respectively, are combined in a network having one input and one output, then the overall transfer function of the network is always of order ≤ n1 +n2 +n3.

1. False, p/q l/m approach and multiple out to keep a constant or positive pole
2. False, it could be stable because of pole zero cancellation
3. False, although it sounds reasonable intuitively, it could be but not necessarily, the poles and zero es get flipped, the zeroes could be outside the unit circle
4. False, it can be, the poles get multiplied and added, if there are cancellations, then the poles could be 2
5. True, same as before
6. True, same reasoning of p/l m/n s/t multiple them out and see that the order is equal to or less than their sums

## 18

## 19

## 20

**21**

Ex 21 ①

1) $a = H_1 u$

$b = H_1^{-1} a = u$

$c = a - b = H_1 u - u$

$d = H_2 u + c = H_2 u + H_1 u - u$

$Y = b - H_2^{-1} d$

$Y = u - H_2^{-1}(H_2 u + H_1 u - u)$

$Y = -H_2^{-1} H_1 u + H_2^{-1} u$

$\frac{Y}{u} = H_2^{-1}(-H_1 + 1)$

$H(s) = \frac{1 - H_1}{H_2}$

## Ex. 21 ②

2)

$$a = H_1 e$$

$$e = U - H_2^{-1} c$$

$$c = a - b = a(1 - H_2)$$

$$b = H_2 a$$

$$y = b - H_1^{-1} c$$

e is the issue with the loop so lets try to find an expression for it to get rid of e.

$$e = U - H_2^{-1}(H_1 e - H_1 H_2 e)$$

$$e = U - H_2^{-1} H_1 e + H_1 e$$ [illegible]

$$e = U + H_1 e(-H_2^{-1} + 1)$$

$$e(1 - H_1(-H_2^{-1} + 1)) = U$$

$$e = \frac{U}{1 - H_1(-H_2^{-1} + 1)}$$

## Ex 21 ③

2 cont) Now, its just substitution and simplification.

$$Y = b - H_1^{-1} c$$

$$Y = H_2 a - H_1^{-1}\left( a(1 - H_2)\right)$$

$$Y = H_2 a - H_1^{-1} a + H_1^{-1} H_2 a$$

$$Y = a\left(H_2 - H_1^{-1} + H_1^{-1} H_2\right)$$

$$Y = e H_1\left( H_2 - H_1^{-1} + H_1^{-1} H_2\right)$$

$$Y = e\left( H_1 H_2 - 1 + H_2\right)$$

$$\frac{Y}{u} = \frac{H_1 H_2 - 1 + H_2}{1 - H_1\left(-H_2^{-1} + 1\right)}$$

Ex 21 (a)

3) $a = U - H_1 b$

$b = H_1^{-1} a$

$Y = H_2 b - b$

Problamatic variable is b lets try to make an expression of b from Y

$Y = b(H_2 - 1)$

$b = \frac{H_2 - 1}{Y}$ I don't like this Y at the bottom lets try to make an expression of b another way.

~~Now, we need an expression that has both Y and U in it.~~

$b = H_1^{-1}(U - H_1 b)$

$b = H_1^{-1} U - b$

$b = \frac{H_1^{-1} U}{2}$

$Y = H_2\left(\frac{H_1^{-1} U}{2}\right) - \frac{H_1^{-1} U}{2}$

factor U out and divide by it.

$\frac{Y}{U} = \frac{H_2 H_1^{-1} - H_1^{-1}}{2}$

Ex 21 (5)

Q4) (1) $a = H_1 U$

(2) $b = H_1^{-1} - a$

(3) $c = H_2^{-1}(U - d)$

(4) $d = a + c + e$

(5) $e = H_2 c$

(6) $Y = b + e$ → problematic. but (4) and (5) have $e$ lets use them to make an expression of $e$.

easy. $= -U$ using (1) and (2).

using (5) $c = H_2^{-1} e$. (7)

using (4)(1) $d = H_1 U + H_2^{-1} e + e$ (8)

using (5), (3) $H_2^{-1} e = H_2^{-1} U - H_2^{-1}(H_1 U + H_2^{-1} e + e)$

$$2 H_2^{-1} e + 2e = \cancel{H_2^{-1}} U - \cancel{H_2^{-1}} H_1 U$$

$e(2H_2^{-1} + 1)$

$$e = \frac{\cancel{H_2^{-1}} U - H_1 U}{H_2^{-1} + 2}$$

Ex 21 6

Q4 cont)
$$Y = -U + \frac{\cancel{H_2} U - H_1 U}{2H_2^{-1} + 1}$$

$$\frac{Y}{U} = -1 + \frac{\cancel{H_2}^{1} - H_1}{\cancel{2} H_2^{-1} + 2}$$

Ex 21

Q5)
$$a = H_1 U + H_2 U$$

no loops, so should be serially possible.

$$b = H_1 U - a = -H_2 U$$

$$c = H_2 U - a = -H_1 U$$

$$d = H_3 a = H_1 H_3 U + H_2 H_3 U$$

$$Y = b + c + d$$

$$Y = U\left(-H_2 - H_1 + H_1 H_3 + H_2 H_3\right)$$

$$\frac{Y}{U} = -H_2 - H_1 + H_3\left(H_1 + H_2\right)$$