Formula sheet Mathematical Modeling

Name	Time domain function	Laplace domain function
Transform pair	$f(t)$	$F(s)$
Superposition	$\alpha_1 f_1(t) + \alpha_2 f_2(t)$	$\alpha_1 F_1(s) + \alpha_2 F_2(s)$
Differentiation	$\frac{d^k}{dt^k} f(t)$	$s^k F(s) - s^{k-1} f(0) - \ldots - \frac{d^{k-1}}{dt^{k-1}} f(0)$
Integration	$\int_0^t f(\tau) d\tau$	$\frac{1}{s} F(s)$
Time scaling	$f(at)$	$\frac{1}{\lvert a \rvert} F\left(\frac{s}{a}\right)$
Time shift	$f(t-\tau)$	$F(s) e^{-s\tau}$
Convolution	$f_1(t) * f_2(t)$	$F_1(s) F_2(s)$
	$= \int_0^t f_1(\tau) f_2(t-\tau) d\tau$	
	$= \int_0^t f_1(t-\tau) f_2(\tau) d\tau$	
Exponential modulation	$e^{-at} f(t)$	$F(s+a)$
Product	$f_1(t) f_2(t)$	$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F_1(\xi) F_2(s-\xi) d\xi$
Multiplication with time	$t f(t)$	$-\frac{d}{ds} F(s)$
Initial value theorem	$f(0+)$	$\lim_{s \to \infty} sF(s)$
Final value theorem	$\lim_{t \to \infty} f(t)$	$\lim_{s \to 0} sF(s)$

Table 1: Properties of the Laplace transform

Name	$f(t)$	$F(s)$
Impulse function	$\delta(t)$	1
Step function	$f(t) = 1$ for $t \geq 0$	$\frac{1}{s}$
Power of t	t^k	$\frac{k!}{s^{k+1}}$
Exponential	e^{-at}	$\frac{1}{s+a}$
Negative exponential	$1 - e^{-at}$	$\frac{a}{s(s+a)}$
Damped power of t	$t^k e^{-at}$	$\frac{k!}{(s+a)^{k+1}}$
Sine	$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
Cosine	$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
Damped sine	$e^{-at} \sin(\omega t)$	$\frac{\omega}{(s+a)^2+\omega^2}$
Damped cosine	$e^{-at} \cos(\omega t)$	$\frac{s+a}{(s+a)^2+\omega^2}$

Table 2: Laplace transform pairs

Degrees	0°	30°	45°	60°	90°	180°	270°	360°
Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
Sine	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	0	-1	0
Cosine	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-1	0	1
Tangent	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undefined	0	undefined	0

Table 3: Angles

$$\operatorname{atan2}(y,x) = \begin{cases} \arctan\left(\frac{y}{x}\right) & \text{if } x > 0 \\ \arctan\left(\frac{y}{x}\right) + \pi & \text{if } x < 0 \text{ and } y \geq 0 \\ \arctan\left(\frac{y}{x}\right) - \pi & \text{if } x < 0 \text{ and } y < 0 \\ \frac{\pi}{2} & \text{if } x = 0 \text{ and } y > 0 \\ -\frac{\pi}{2} & \text{if } x = 0 \text{ and } y < 0 \\ \text{undefined} & \text{if } x = 0 \text{ and } y = 0 \end{cases}$$

Table 4: PoleCoordinates

Name	Time domain function	z-domain function
Transform pair	$f[k]$	$F(z)$
Superposition	$\alpha_1 f_1[k] + \alpha_2 f_2[k]$	$\alpha_1 F_1(z) + \alpha_2 F_2(z)$
Time delay	$f[k-n]$	$z^{-n}\left(F(z) + \sum_{i=1}^{n} f[-i]z^{i}\right)$
Time advance	$f[k+n]$	$z^{n}\left(F(z) + \sum_{i=1}^{n} f[i]z^{-i}\right)$
Convolution	$f_1[k] * f_2[k]$	$F_1(z)F_2(z)$
	$= \sum_{i=0}^{k} f_1[k-i]f_2[i]$	
	$= \sum_{i=0}^{k} f_1[i]f_2[k-i]$	
Frequency scaling	$b^k f[k]$	$F\left(\frac{z}{b}\right)$
Complex modulation	$e^{ak} f[k]$	$F(ze^{-a})$
Power modulation	$a^k f[k]$	$F(a^{-1}z)$
Initial value theorem	$f[0]$	$\lim\limits_{\lvert z\rvert\to\infty} F(z)$
Final value theorem	$\lim\limits_{k\to\infty} f[k]$	$\lim\limits_{z\to 1}(z-1)F(z)$

Table 5: Properties of the z-transform

Name	$f[k]$	$F(z)$
Impulse function	$\delta[k] = \left\{ \begin{array}{ll} 1, & k=0 \\ 0 & k \neq 0 \end{array} \right\}$	1
Step function	$f[k] = 1$ for $k \geq 0$	$\frac{z}{z-1} = \frac{1}{1-z^{-1}}$
Power	b^k	$\frac{z}{z-b} = \frac{1}{1-bz^{-1}}$
Exponential	e^{ak}	$\frac{z}{z-e^{a}}$
Ramp sequence	k	$\frac{z}{(z-1)^2}$
Parabolic sequence	k^2	$\frac{z(z+1)}{(z-1)^3}$
Power ramp	$b^k k$	$\frac{bz}{(z-b)^2}$
General power	$\binom{k}{j-1} b^{k-(j-1)}$	$\frac{z}{(z-b)^j}$
Exponential ramp	$e^{ak} k$	$\frac{ze^{a}}{(z-e^{a})^2}$
Sine	$\sin[\omega k]$	$\frac{\sin(a)z}{z^2-2\cos(\omega)z+1}$
Cosine	$\cos[\omega k]$	$\frac{z(z-\cos(a))}{z^2-2\cos(\omega)z+1}$
Power sine	$b^k \sin[\omega k]$	$\frac{b\sin(\omega)z}{z^2-2b\cos(a)z+b^2}$
Power cosine	$b^k \cos[\omega k]$	$\frac{z(z-b\cos(\omega))}{z^2-2b\cos(a)z+b^2}$

Table 6: z-transform pairs

Continuous-time	Discrete-time
$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}u(t)$	$\mathbf{x}[k+1] = \mathbf{A}\mathbf{x}[k] + \mathbf{b}u[k]$
$y(t) = \mathbf{c}\mathbf{x}(t) + du(t)$	$y[k] = \mathbf{c}\mathbf{x}[k] + du[k]$
$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{b}u(\tau)d\tau$	$\mathbf{x}[k] = \mathbf{A}^k\mathbf{x}[0] + \sum_{j=1}^{k-1}\mathbf{A}^{k-j-1}\mathbf{b}u[j]$
$H(s) = \mathbf{c}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{b} + d$	$H(z) = \mathbf{c}(z\mathbf{I}-\mathbf{A})^{-1}\mathbf{b} + d$
$h(t) = \mathbf{c}e^{\mathbf{A}t}\mathbf{b} + d\delta(t)$	$h[k] = \begin{cases} d & k=0 \\ \mathbf{c}\mathbf{A}^{k-1}\mathbf{b} & k>0 \end{cases}$
$h(t) = \mathfrak{L}^{-1}\{H(s)\} = \sum_i A_i e^{p_i t} + \sum_r \sum_{j=1}^{m_r} B_{r,j} \frac{t^{(j-1)}}{(j-1)!} e^{\tilde{p}_r t}$ (PFE)	$h[k] = \mathfrak{z}^{-1}\{H(z)\} = \sum_i A_i p_i^k + \sum_r \sum_{j=1}^{m_r} B_{r,j} \binom{k}{j-1} \tilde{p}_r^{k-(j-1)}$. (PFE)
$B_{r,j} = \frac{1}{(m_r-j)!}\left\{\frac{d^{m_r-j}}{ds^{m_r-j}}(s-\tilde{p}_r)^{m_r}h(s)\right\}\vert_{s=\tilde{p}_r}$	$B_{r,j} = \frac{1}{(m_r-j)!}\left(\frac{d^{m_r-j}}{dz^{m_r-j}}\frac{(z-\tilde{p}_r)^{m_r}H(z)}{z}\right)\vert_{z=\tilde{p}_r}$
$(s\mathbf{I}-\mathbf{A})^{-1} \underset{\mathfrak{L}}{\overset{\mathfrak{L}^{-1}}{\rightleftarrows}} e^{\mathbf{A}t}$	$(z\mathbf{I}-\mathbf{A})^{-1} \underset{\mathfrak{z}}{\overset{\mathfrak{z}^{-1}}{\rightleftarrows}} \mathbf{A}^{k-1}$
If $\bar{\mathbf{x}}(t) = \mathbf{T}\mathbf{x}(t)$ then	If $\bar{\mathbf{x}}[k] = \mathbf{T}\mathbf{x}[k]$ then
$(\mathbf{A},\mathbf{b},\mathbf{c},d) \xrightarrow{T} (\bar{\mathbf{A}},\bar{\mathbf{b}},\bar{\mathbf{c}},\bar{d})$ where	$(\mathbf{A},\mathbf{b},\mathbf{c},d) \xrightarrow{T} (\bar{\mathbf{A}},\bar{\mathbf{b}},\bar{\mathbf{c}},\bar{d})$ where
$\bar{\mathbf{A}} = \mathbf{T}\mathbf{A}\mathbf{T}^{-1}$	$\bar{\mathbf{A}} = \mathbf{T}\mathbf{A}\mathbf{T}^{-1}$
$\bar{\mathbf{b}} = \mathbf{T}\mathbf{b}$	$\bar{\mathbf{b}} = \mathbf{T}\mathbf{b}$
$\bar{\mathbf{c}} = \mathbf{c}\mathbf{T}^{-1}$	$\bar{\mathbf{c}} = \mathbf{c}\mathbf{T}^{-1}$
$\bar{d} = d$	$\bar{d} = d$
$u(t) = \sin(\omega t)$,	$u[k] = \sin(\omega kT)$,
$y_s(t) = \vert G(i\omega)\vert \sin(\omega t + \angle G(i\omega))$	$y_s[k] = \vert G(e^{i\omega T})\vert \sin\left(\omega kT + \angle G(e^{i\omega T})\right)$

Table 7: Summary of continuous-time and discrete-time systems theory

	Observable canonical form	Controllable canonical form
$\mathbf{A}$	$\begin{pmatrix} -a_1 & 1 & 0 & 0 & \dots & 0 & 0 \\ -a_2 & 0 & 1 & 0 & \dots & 0 & 0 \\ -a_3 & 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -a_{n-2} & 0 & 0 & 0 & \dots & 1 & 0 \\ -a_{n-1} & 0 & 0 & 0 & \dots & 0 & 1 \\ -a_n & 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} -a_1 & -a_2 & \dots & -a_{n-1} & -a_n \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}$
$\mathbf{b}$	$\begin{pmatrix} b_1 - a_1 b_0 \\ b_2 - a_2 b_0 \\ \vdots \\ b_n - a_n b_0 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$
$\mathbf{c}$	$\begin{pmatrix} 1 & 0 & \dots & 0 \end{pmatrix}$	$\begin{pmatrix} b_1 - a_1 b_0 & b_2 - a_2 b_0 & \dots & b_n - a_n b_0 \end{pmatrix}$
d	b_0	b_0

Table 8: Canonical forms

$\mathbf{S}_1 = \mathbf{I}$	$a_1 = -\mathrm{tr}(\mathbf{A})$
$\mathbf{S}_2 = \mathbf{S}_1\mathbf{A} + a_1\mathbf{I}$	$a_2 = -\frac{1}{2}\mathrm{tr}(\mathbf{S}_2\mathbf{A})$
$\mathbf{S}_3 = \mathbf{S}_2\mathbf{A} + a_2\mathbf{I}$	$a_3 = -\frac{1}{3}\mathrm{tr}(\mathbf{S}_3\mathbf{A})$
$\vdots$	$\vdots$
$\mathbf{S}_n = \mathbf{S}_{n-1}\mathbf{A} + a_{n-1}\mathbf{I}$	$a_n = -\frac{1}{n}\mathrm{tr}(\mathbf{S}_n\mathbf{A})$
$(s\mathbf{I}-\mathbf{A})^{-1} = \frac{\mathbf{S}_1 s^{n-1} + \mathbf{S}_2 s^{n-2} + \ldots + \mathbf{S}_n}{s^n + a_1 s^{n-1} + \ldots + a_n}$	

Table 9: Fadeev algorithm

Parallel interconnection	Serial interconnection
$\mathbf{z}[k+1] = \begin{pmatrix} \mathbf{A}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_2 \end{pmatrix} \mathbf{z}[k] + \begin{pmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{pmatrix} r[k]$	$\mathbf{z}[k+1] = \begin{pmatrix} \mathbf{A}_1 & \mathbf{0} \\ \mathbf{b}_2\mathbf{c}_1 & \mathbf{A}_2 \end{pmatrix} \mathbf{z}[k] + \begin{pmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 d_1 \end{pmatrix} r[k]$
$y[k] = \begin{pmatrix} \mathbf{c}_1 & \mathbf{c}_2 \end{pmatrix} z[k] + (d_1 + d_2) r[k]$	$y[k] = \begin{pmatrix} d_2\mathbf{c}_1 & \mathbf{c}_2 \end{pmatrix} z[k] + (d_2 d_1) r[k]$

Table 10: State-space interconnections

$\mathbf{x}[k+1] = \mathbf{\Phi}\mathbf{x}[k] + \mathbf{\Gamma}u[k]$

$y[k] = \mathbf{c}\mathbf{x}[k] + du[k]$

$\mathbf{\Phi} = e^{\mathbf{A}T}$

$\mathbf{\Gamma} = \int_0^T e^{\mathbf{A}\gamma}\mathbf{b}d\gamma$

$\mathbf{M} = \begin{pmatrix} \mathbf{A} & \mathbf{b} \\ \mathbf{0} & 0 \end{pmatrix}$

$e^{\mathbf{M}T} = \begin{pmatrix} \mathbf{\Phi} & \mathbf{\Gamma} \\ \mathbf{0} & 1 \end{pmatrix}$

$$\begin{pmatrix} e^{\lambda_1 T} \\ e^{\lambda_2 T} \\ \vdots \\ e^{\lambda_n T} \end{pmatrix} = \begin{pmatrix} \lambda_1^{n-1} & \lambda_1^{n-2} & \dots & 1 \\ \lambda_2^{n-1} & \lambda_2^{n-2} & \dots & 1 \\ \vdots & \vdots & & \vdots \\ \lambda_n^{n-1} & \lambda_n^{n-2} & \dots & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}$$

In case of repeated eigenvalues λ_i, for each repetition differentiate lhs and rhs w.r.t. λ_i.
In case of complex values split it up, where you take the real part of both lhs and rhs and another one where you have both imaginary parts.

Table 11: Discrete-time equivalents of continuous-time systems

$\mathbf{W}_c = \begin{pmatrix} \mathbf{B} & \mathbf{AB} & \dots & \mathbf{A}^{n-1}\mathbf{B} \end{pmatrix}$

$$\bar{\mathbf{W}}_c^{-1} = \begin{pmatrix} 1 & a_1 & a_2 & \dots & a_{n-2} & a_{n-1} \\ 0 & 1 & a_1 & \ddots & a_{n-3} & a_{n-2} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & a_1 & a_2 \\ 0 & \dots & 0 & 0 & 1 & a_1 \\ 0 & \dots & 0 & 0 & 0 & 1 \end{pmatrix}$$

$\mathbf{T}^{-1} = \mathbf{W}_c\bar{\mathbf{W}}_c^{-1}$

$\mathbf{u}[k] = -\bar{L}\bar{\mathbf{x}}[k] = -\bar{L}\mathbf{T}\mathbf{x}[k] = -L\mathbf{x}[k]$

$\bar{L} = \begin{pmatrix} \bar{l}_1 & \bar{l}_2 & \dots & \bar{l}_n \end{pmatrix} = \begin{pmatrix} \tilde{a}_1 - a_1 & \tilde{a}_2 - a_2 & \dots & \tilde{a}_n - a_n \end{pmatrix}$

Table 12: Control