Exam Mathematical Modeling 01/04/2016

- This exam is **Closed book**. You are allowed to use a non-programmable, non-graphical (hence a simple, non cheating) calculator.
- Make sure that your cell phone, tablet, laptop, etc. is on the other side of the room and turned off
- for this exam you can get 100 points. Your grade will be the number of points earned divided by 10. Additionally, you can have up to 0.5 bonus points from the assignment during the practicals.
- Good luck!

1. (6 points) In a stretch of savanna (where food ain't plentiful) there lives a heard of buffaloes that eat grass, a heard of giraffes that eat leaves from the trees and a group of lions that tend to eat buffaloes, but when given the opportunity will not refrain from eating giraffes. Specify a dynamical model, i.e. give the equations involving some unspecified positive parameters, for this situation. Obviously, we are expecting equations/a model that fits in the framework as discussed in the lectures. Motivate why you defined the model like this.

2. Consider a linear input-output system in continuous-time given by a third order differential equation of the form:

$$y^{(3)}(t) + 5y^{(2)}(t) - u^{(2)}(t) + 9y^{(1)}(t) - u^{(1)}(t) + 5y(t) + 6u(t) = 0$$

   (a) (6 points) Compute the transfer function $H(s)$ of the system.

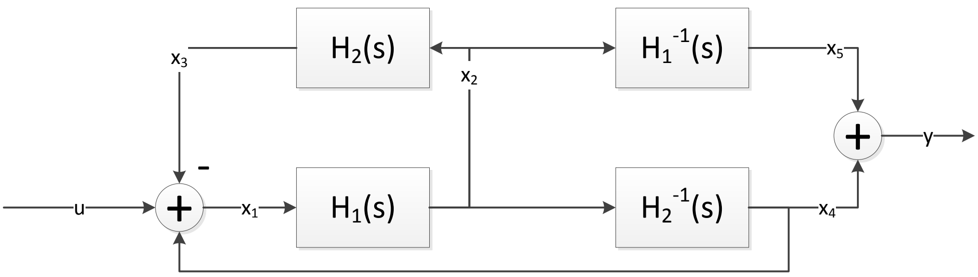
   (b) (6 points) Determine the poles of the system. Hint: one pole is located at $s = -1$. What does this mean for stability of the system? (Explain your answer!)

   (c) (15 points) Determine the time-domain output $y(t)$ of the system for the input $u(t) = e^t$ given that the system starts completely at rest at $t = 0$ (zero initial conditions)

3. (a) (6 points) A sinusoid $u(t) = 3\sin(2t)$ is used as as an input to a system $H(s) = \frac{s^2-5s+3}{s^3+5s^2+8s+4}$. Compute the exact frequency response to the frequency matching the input. Don't leave complex numbers in the denominator.

   (b) (11 points) Given the aforementioned input $u(t) = 3\sin(2t)$, compute the steady-state time-domain output $y_s(t)$. If you don't have a calculator just specify the angle in terms of the appropriate trigonometric function. You are allowed to give a numerical approximation.

   (c) (15 points) Consider the diagram below.



   Compute the transfer function $H(s)$ of the entire network in terms of the transfer functions $H_1(s)$ and $H_2(s)$.

4. Consider the discrete-time system, given by the state-space representation

$$x[k+1] = \begin{pmatrix} 8 & 24 & 0 \\ \frac{1}{2} & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix} x[k] + \begin{pmatrix} \frac{1}{4} \\ 0 \\ 0 \end{pmatrix} u[k]$$
$$y[k] = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} x[k] + u[k]$$

(a) (15 points) Compute the transfer function $H(z)$ of the system.

(b) (20 points) Determine a state feedback law $u[k] = -Lx[k]$ such that the poles of the resulting closed loop system are located at $-\frac{1}{2}, 0$ and $\frac{1}{4}$ respectively.

Formula sheet Mathematical Modeling

Name	Time domain function	Laplace domain function
Transform pair	$f(t)$	$F(s)$
Superposition	$\alpha_1 f_1(t) + \alpha_2 f_2(t)$	$\alpha_1 F_1(s) + \alpha_2 F_2(s)$
Differentiation	$\frac{d^k}{dt^k} f(t)$	$s^k F(s) - s^{k-1} f(0) - \ldots - \frac{d^{k-1}}{dt^{k-1}} f(0)$
Integration	$\int_0^t f(\tau) d\tau$	$\frac{1}{s} F(s)$
Time scaling	$f(at)$	$\frac{1}{\lvert a \rvert} F\left(\frac{s}{a}\right)$
Time shift	$f(t-\tau)$	$F(s) e^{-s\tau}$
Convolution	$f_1(t) * f_2(t)$	$F_1(s) F_2(s)$
	$= \int_0^t f_1(\tau) f_2(t-\tau) d\tau$	
	$= \int_0^t f_1(t-\tau) f_2(\tau) d\tau$	
Exponential modulation	$e^{-at} f(t)$	$F(s+a)$
Product	$f_1(t) f_2(t)$	$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F_1(\xi) F_2(s-\xi) d\xi$
Multiplication with time	$t f(t)$	$-\frac{d}{ds} F(s)$
Initial value theorem	$f(0+)$	$\lim\limits_{s \to \infty} sF(s)$
Final value theorem	$\lim\limits_{t \to \infty} f(t)$	$\lim\limits_{s \to 0} sF(s)$

Table 1: Properties of the Laplace transform

Name	$f(t)$	$F(s)$
Impulse function	$\delta(t)$	1
Step function	$f(t) = 1$ for $t \geq 0$	$\frac{1}{s}$
Power of t	t^k	$\frac{k!}{s^{k+1}}$
Exponential	e^{-at}	$\frac{1}{s+a}$
Negative exponential	$1 - e^{-at}$	$\frac{a}{s(s+a)}$
Damped power of t	$t^k e^{-at}$	$\frac{k!}{(s+a)^{k+1}}$
Sine	$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
Cosine	$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
Damped sine	$e^{-at} \sin(\omega t)$	$\frac{\omega}{(s+a)^2+\omega^2}$
Damped cosine	$e^{-at} \cos(\omega t)$	$\frac{s+a}{(s+a)^2+\omega^2}$

Table 2: Laplace transform pairs

Degrees	$0°$	$30°$	$45°$	$60°$	$90°$	$180°$	$270°$	$360°$
Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
Sine	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	0	-1	0
Cosine	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-1	0	1
Tangent	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undefined	0	undefined	0

Table 3: Angles

$$\operatorname{atan2}(y,x) = \begin{cases} \arctan\left(\frac{y}{x}\right) & \text{if } x > 0 \\ \arctan\left(\frac{y}{x}\right) + \pi & \text{if } x < 0 \text{ and } y \geq 0 \\ \arctan\left(\frac{y}{x}\right) - \pi & \text{if } x < 0 \text{ and } y < 0 \\ \frac{\pi}{2} & \text{if } x = 0 \text{ and } y > 0 \\ -\frac{\pi}{2} & \text{if } x = 0 \text{ and } y < 0 \\ \text{undefined} & \text{if } x = 0 \text{ and } y = 0 \end{cases}$$

Table 4: PoleCoordinates

Name	Time domain function	z-domain function
Transform pair	$f[k]$	$F(z)$
Superposition	$\alpha_1 f_1[k] + \alpha_2 f_2[k]$	$\alpha_1 F_1(z) + \alpha_2 F_2(z)$
Time delay	$f[k-n]$	$z^{-n}\left(F(z) + \sum_{i=1}^{n} f[-i]z^{i}\right)$
Time advance	$f[k+n]$	$z^{n}\left(F(z) + \sum_{i=1}^{n} f[i]z^{-i}\right)$
Convolution	$f_1[k] * f_2[k]$	$F_1(z)F_2(z)$
	$= \sum_{i=0}^{k} f_1[k-i]f_2[i]$	
	$= \sum_{i=0}^{k} f_1[i]f_2[k-i]$	
Frequency scaling	$b^k f[k]$	$F\left(\frac{z}{b}\right)$
Complex modulation	$e^{ak} f[k]$	$F(ze^{-a})$
Power modulation	$a^k f[k]$	$F(a^{-1}z)$
Initial value theorem	$f[0]$	$\lim_{\lvert z\rvert\to\infty} F(z)$
Final value theorem	$\lim_{k\to\infty} f[k]$	$\lim_{z\to 1}(z-1)F(z)$

Table 5: Properties of the z-transform

Name	$f[k]$	$F(z)$
Impulse function	$\delta[k] = \left\{ \begin{array}{cc} 1, & k = 0 \\ 0 & k \neq 0 \end{array} \right\}$	1
Step function	$f[k] = 1$ for $k \geq 0$	$\frac{z}{z-1} = \frac{1}{1-z^{-1}}$
Power	b^k	$\frac{z}{z-b} = \frac{1}{1-bz^{-1}}$
Exponential	e^{ak}	$\frac{z}{z-e^a}$
Ramp sequence	k	$\frac{z}{(z-1)^2}$
Parabolic sequence	k^2	$\frac{z(z+1)}{(z-1)^3}$
Power ramp	$b^k k$	$\frac{bz}{(z-b)^2}$
General power	$\binom{k}{j-1} b^{k-(j-1)}$	$\frac{z}{(z-b)^j}$
Exponential ramp	$e^{ak}k$	$\frac{ze^a}{(z-e^a)^2}$
Sine	$\sin[\omega k]$	$\frac{\sin(a)z}{z^2-2\cos(\omega)z+1}$
Cosine	$\cos[\omega k]$	$\frac{z(z-\cos(a))}{z^2-2\cos(\omega)z+1}$
Power sine	$b^k \sin[\omega k]$	$\frac{b\sin(\omega)z}{z^2-2b\cos(a)z+b^2}$
Power cosine	$b^k \cos[\omega k]$	$\frac{z(z-b\cos(\omega))}{z^2-2b\cos(a)z+b^2}$

Table 6: z-transform pairs

Continuous-time	Discrete-time
$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}u(t)$	$\mathbf{x}[k+1] = \mathbf{A}\mathbf{x}[k] + \mathbf{b}u[k]$
$y(t) = \mathbf{c}\mathbf{x}(t) + du(t)$	$y[k] = \mathbf{c}\mathbf{x}[k] + du[k]$
$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{b}u(\tau)d\tau$	$\mathbf{x}[k] = \mathbf{A}^k\mathbf{x}[0] + \sum_{j=1}^{k-1}\mathbf{A}^{k-j-1}\mathbf{b}u[j]$
$H(s) = \mathbf{c}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{b} + d$	$H(z) = \mathbf{c}(z\mathbf{I}-\mathbf{A})^{-1}\mathbf{b} + d$
$h(t) = \mathbf{c}e^{\mathbf{A}t}\mathbf{b} + d\delta(t)$	$h[k] = \begin{cases} d & k=0 \\ \mathbf{c}\mathbf{A}^{k-1}\mathbf{b} & k>0 \end{cases}$
$h(t) = \mathfrak{L}^{-1}\{H(s)\} = \sum_i A_i e^{p_i t} + \sum_r \sum_{j=1}^{m_r} B_{r,j}\frac{t^{(j-1)}}{(j-1)!}e^{\tilde{p}_r t}$ (PFE)	$h[k] = \mathfrak{z}^{-1}\{H(z)\} = \sum_i A_i p_i^k + \sum_r \sum_{j=1}^{m_r} B_{r,j}\begin{pmatrix} k \\ j-1 \end{pmatrix}\tilde{p}_r^{k-(j-1)}$. (PFE)
$B_{r,j} = \frac{1}{(m_r-j)!}\left\{\frac{d^{m_r-j}}{ds^{m_r-j}}(s-\tilde{p}_r)^{m_r}h(s)\right\}\|_{s=\tilde{p}_r}$	$B_{r,j} = \frac{1}{(m_r-j)!}\left(\frac{d^{m_r-j}}{dz^{m_r-j}}\frac{(z-\tilde{p}_r)^{m_r}H(z)}{z}\right)\|_{z=\tilde{p}_r}$
$(s\mathbf{I}-\mathbf{A})^{-1} \underset{\mathfrak{L}}{\overset{\mathfrak{L}^{-1}}{\rightleftarrows}} e^{\mathbf{A}t}$	$(z\mathbf{I}-\mathbf{A})^{-1} \underset{\mathfrak{z}}{\overset{\mathfrak{z}^{-1}}{\rightleftarrows}} \mathbf{A}^{k-1}$
If $\bar{\mathbf{x}}(t) = \mathbf{T}\mathbf{x}(t)$ then	If $\bar{\mathbf{x}}[k] = \mathbf{T}\mathbf{x}[k]$ then
$(\mathbf{A},\mathbf{b},\mathbf{c},d) \xrightarrow{T} (\bar{\mathbf{A}},\bar{\mathbf{b}},\bar{\mathbf{c}},\bar{d})$ where	$(\mathbf{A},\mathbf{b},\mathbf{c},d) \xrightarrow{T} (\bar{\mathbf{A}},\bar{\mathbf{b}},\bar{\mathbf{c}},\bar{d})$ where
$\bar{\mathbf{A}} = \mathbf{T}\mathbf{A}\mathbf{T}^{-1}$	$\bar{\mathbf{A}} = \mathbf{T}\mathbf{A}\mathbf{T}^{-1}$
$\bar{\mathbf{b}} = \mathbf{T}\mathbf{b}$	$\bar{\mathbf{b}} = \mathbf{T}\mathbf{b}$
$\bar{\mathbf{c}} = \mathbf{c}\mathbf{T}^{-1}$	$\bar{\mathbf{c}} = \mathbf{c}\mathbf{T}^{-1}$
$\bar{d} = d$	$\bar{d} = d$
$u(t) = \sin(\omega t)$,	$u[k] = \sin(\omega kT)$,
$y_s(t) = \|G(i\omega)\|\sin(\omega t + \angle G(i\omega))$	$y_s[k] = \|G(e^{i\omega T})\|\sin\left(\omega kT + \angle G(e^{i\omega T})\right)$

Table 7: Summary of continuous-time and discrete-time systems theory

	Observable canonical form	Controllable canonical form
$\mathbf{A}$	$\begin{pmatrix} -a_1 & 1 & 0 & 0 & \dots & 0 & 0 \\ -a_2 & 0 & 1 & 0 & \dots & 0 & 0 \\ -a_3 & 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -a_{n-2} & 0 & 0 & 0 & \dots & 1 & 0 \\ -a_{n-1} & 0 & 0 & 0 & \dots & 0 & 1 \\ -a_n & 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} -a_1 & -a_2 & \dots & -a_{n-1} & -a_n \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}$
$\mathbf{b}$	$\begin{pmatrix} b_1 - a_1 b_0 \\ b_2 - a_2 b_0 \\ \vdots \\ b_n - a_n b_0 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$
$\mathbf{c}$	$\begin{pmatrix} 1 & 0 & \dots & 0 \end{pmatrix}$	$\begin{pmatrix} b_1 - a_1 b_0 & b_2 - a_2 b_0 & \dots & b_n - a_n b_0 \end{pmatrix}$
d	b_0	b_0

Table 8: Canonical forms

$\mathbf{S}_1 = \mathbf{I}$	$a_1 = -\mathrm{tr}(\mathbf{A})$
$\mathbf{S}_2 = \mathbf{S}_1\mathbf{A} + a_1\mathbf{I}$	$a_2 = -\frac{1}{2}\mathrm{tr}(\mathbf{S}_2\mathbf{A})$
$\mathbf{S}_3 = \mathbf{S}_2\mathbf{A} + a_2\mathbf{I}$	$a_3 = -\frac{1}{3}\mathrm{tr}(\mathbf{S}_3\mathbf{A})$
$\vdots$	$\vdots$
$\mathbf{S}_n = \mathbf{S}_{n-1}\mathbf{A} + a_{n-1}\mathbf{I}$	$a_n = -\frac{1}{n}\mathrm{tr}(\mathbf{S}_n\mathbf{A})$
$(s\mathbf{I}-\mathbf{A})^{-1} = \frac{\mathbf{S}_1 s^{n-1} + \mathbf{S}_2 s^{n-2} + \ldots + \mathbf{S}_n}{s^n + a_1 s^{n-1} + \ldots + a_n}$	

Table 9: Fadeev algorithm

Parallel interconnection	Serial interconnection
$\mathbf{z}[k+1] = \begin{pmatrix} \mathbf{A}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_2 \end{pmatrix} \mathbf{z}[k] + \begin{pmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{pmatrix} r[k]$	$\mathbf{z}[k+1] = \begin{pmatrix} \mathbf{A}_1 & \mathbf{0} \\ \mathbf{b}_2\mathbf{c}_1 & \mathbf{A}_2 \end{pmatrix} \mathbf{z}[k] + \begin{pmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 d_1 \end{pmatrix} r[k]$
$y[k] = \begin{pmatrix} \mathbf{c}_1 & \mathbf{c}_2 \end{pmatrix} z[k] + (d_1 + d_2) r[k]$	$y[k] = \begin{pmatrix} d_2\mathbf{c}_1 & \mathbf{c}_2 \end{pmatrix} z[k] + (d_2 d_1) r[k]$

Table 10: State-space interconnections

$\mathbf{x}[k+1] = \mathbf{\Phi}\mathbf{x}[k] + \mathbf{\Gamma}u[k]$

$y[k] = \mathbf{c}\mathbf{x}[k] + du[k]$

$\mathbf{\Phi} = e^{\mathbf{A}T}$

$\mathbf{\Gamma} = \int_0^T e^{\mathbf{A}\gamma}\mathbf{b}d\gamma$

$\mathbf{M} = \begin{pmatrix} \mathbf{A} & \mathbf{b} \\ \mathbf{0} & 0 \end{pmatrix}$

$e^{\mathbf{M}T} = \begin{pmatrix} \mathbf{\Phi} & \mathbf{\Gamma} \\ \mathbf{0} & 1 \end{pmatrix}$

$$\begin{pmatrix} e^{\lambda_1 T} \\ e^{\lambda_2 T} \\ \vdots \\ e^{\lambda_n T} \end{pmatrix} = \begin{pmatrix} \lambda_1^{n-1} & \lambda_1^{n-2} & \dots & 1 \\ \lambda_2^{n-1} & \lambda_2^{n-2} & \dots & 1 \\ \vdots & \vdots & & \vdots \\ \lambda_n^{n-1} & \lambda_n^{n-2} & \dots & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}$$

In case of repeated eigenvalues λ_i, for each repetition differentiate lhs and rhs w.r.t. λ_i.
In case of complex values split it up, where you take the real part of both lhs and rhs and another one where you have both imaginary parts.

Table 11: Discrete-time equivalents of continuous-time systems

$\mathbf{W}_c = \begin{pmatrix} \mathbf{B} & \mathbf{AB} & \dots & \mathbf{A}^{n-1}\mathbf{B} \end{pmatrix}$

$$\bar{\mathbf{W}}_c^{-1} = \begin{pmatrix} 1 & a_1 & a_2 & \dots & a_{n-2} & a_{n-1} \\ 0 & 1 & a_1 & \ddots & a_{n-3} & a_{n-2} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & a_1 & a_2 \\ 0 & \dots & 0 & 0 & 1 & a_1 \\ 0 & \dots & 0 & 0 & 0 & 1 \end{pmatrix}$$

$\mathbf{T}^{-1} = \mathbf{W}_c\bar{\mathbf{W}}_c^{-1}$

$\mathbf{u}[k] = -\bar{L}\bar{\mathbf{x}}[k] = -\bar{L}\mathbf{T}\mathbf{x}[k] = -L\mathbf{x}[k]$

$\bar{L} = \begin{pmatrix} \bar{l}_1 & \bar{l}_2 & \dots & \bar{l}_n \end{pmatrix} = \begin{pmatrix} \tilde{a}_1 - a_1 & \tilde{a}_2 - a_2 & \dots & \tilde{a}_n - a_n \end{pmatrix}$

Table 12: Control