Exam Mathematical Modelling 2016-2017

- Monday April 3rd 2017, 9.00-12.00h, Tapijn Z
- This exam is **Closed book**. You are allowed to use a non-graphical (hence a simple, non cheating) calculator.
- Make sure that your cell phone, tablet, laptop, watch, etc. is on the other side of the room and that your cell phone is turned off
- for this exam you can get 100 points. Your grade will be the number of points earned divided by 10. Additionally, you can have up to 0.5 bonus points from the assignment during the labs.
- This exam consists of four questions
- This exam, including formula sheets, consists of six pages
- Good luck!

1. True/false questions
   Indicate for each of the propositions whether it is true or false and *explain why*.
   (a) (5 points) Minimality
   Given the discrete-time LTI state-space system

$$\begin{aligned}\mathbf{x}[k+1] &= \begin{pmatrix} 1 & 0 & 3 \\ -1 & 0 & -3 \\ 0 & 0 & 1 \end{pmatrix} \mathbf{x}[k] + \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} u[k] \\ y[k] &= \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \mathbf{x}[k].\end{aligned}$$

   This system is minimal

   (b) (5 points) Discrete-time equivalents of continuous-time systems
   If for a stable continuous-time LTI state-space system with piecewise constant inputs, corresponding with some sampling time $T > 0$ a discrete-time equivalent system is computed, then the resulting discrete-time system is stable.

   (c) (5 points) On Malta there is a large population x of bunnies. In the hypothetical case that someone introduces a population of killer bunnies y that do eat the same food as the regular bunnies, but tear any other bunny that they encounter apart without eating them, then an appropriate Lotka-Volterra model for the population sizes is:

$$\begin{aligned}x'(t) &= (\epsilon_1 - \sigma_1 x_1(t) - \alpha_1 y(t))\, x(t) \\ y'(t) &= (\epsilon_2 - \sigma_2 y_1(t) - \alpha_2 x(t))\, y(t),\end{aligned}$$

   with $\epsilon_1, \epsilon_2, \sigma_1, \sigma_2, \alpha_1, \alpha_2 \geq 0$.

2. Given the discrete-time LTI state-space system

$$\begin{aligned}\mathbf{x}[k+1] &= \begin{pmatrix} 1 & -\frac{1}{2} & 0 \\ \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \end{pmatrix} \mathbf{x}[k] + \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} u[k] \\ y[k] &= \begin{pmatrix} \frac{1}{4} & -\frac{1}{2} & 2 \end{pmatrix} \mathbf{x}[k].\end{aligned}$$

   (a) (18 points) Compute the transfer function $H(z)$ of this system.
   (b) (5 points) Check whether the system is stable or not.

3. Consider a continuous-system governed by the following transfer function:

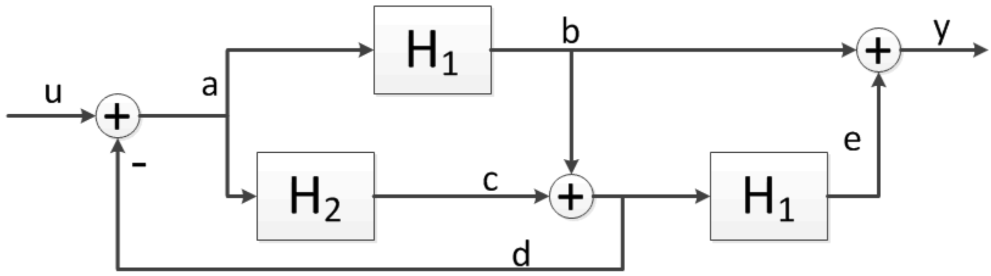
$$H(s) = \frac{s^2 + 2s}{(s+3)(s+1+i)(s+1-i)}.$$

(a) (15 points) Find the impulse response $h(t)$ of this system.

(b) (8 points) Given that an input $u(t) = 4\cos(t)$ is given to the system. Compute the *exact* matching frequency response of the system.

(c) (8 points) Compute the steady state output $y_s(t)$ corresponding to the aforementioned input $u(t)$.

(d) (10 points) Given the network interconnection in the following diagram:



Compute the overall transfer function $H(s)$ in terms of H_1 and H_2.

4. (a) (6 points) Given the following differential equation:

$$y^{(3)}(t) - u^{(3)}(t) + 6u^{(2)}(t) + 7y^{(2)}(t) = 11u^{(1)}(t) - 16y^{(1)}(t) - 12y(t) - 6u(t).$$

Find the transfer function of the system.

(b) (5 points) Provide a state-space description for this system.

(c) (10 points) One wants to find a state-feedback such that all poles of the closed loop system are at -4. Provide the desired state-feedback L.

Formula sheet Mathematical Modeling

Name	Time domain function	Laplace domain function
Transform pair	$f(t)$	$F(s)$
Superposition	$\alpha_1 f_1(t) + \alpha_2 f_2(t)$	$\alpha_1 F_1(s) + \alpha_2 F_2(s)$
Differentiation	$\frac{d^k}{dt^k} f(t)$	$s^k F(s) - s^{k-1} f(0) - \ldots - \frac{d^{k-1}}{dt^{k-1}} f(0)$
Integration	$\int_0^t f(\tau) d\tau$	$\frac{1}{s} F(s)$
Time scaling	$f(at)$	$\frac{1}{\lvert a \rvert} F\left(\frac{s}{a}\right)$
Time shift	$f(t-\tau)$	$F(s)e^{-s\tau}$
Convolution	$f_1(t) * f_2(t)$	$F_1(s)F_2(s)$
	$= \int_0^t f_1(\tau) f_2(t-\tau) d\tau$	
	$= \int_0^t f_1(t-\tau) f_2(\tau) d\tau$	
Exponential modulation	$e^{-at} f(t)$	$F(s+a)$
Product	$f_1(t) f_2(t)$	$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F_1(\xi) F_2(s-\xi) d\xi$
Multiplication with time	$tf(t)$	$-\frac{d}{ds} F(s)$
Initial value theorem	$f(0+)$	$\lim_{s \to \infty} sF(s)$
Final value theorem	$\lim_{t \to \infty} f(t)$	$\lim_{s \to 0} sF(s)$

Table 1: Properties of the Laplace transform

Name	$f(t)$	$F(s)$
Impulse function	$\delta(t)$	1
Step function	$f(t) = 1$ for $t \geq 0$	$\frac{1}{s}$
Power of t	t^k	$\frac{k!}{s^{k+1}}$
Exponential	e^{-at}	$\frac{1}{s+a}$
Negative exponential	$1 - e^{-at}$	$\frac{a}{s(s+a)}$
Damped power of t	$t^k e^{-at}$	$\frac{k!}{(s+a)^{k+1}}$
Sine	$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
Cosine	$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
Damped sine	$e^{-at} \sin(\omega t)$	$\frac{\omega}{(s+a)^2+\omega^2}$
Damped cosine	$e^{-at} \cos(\omega t)$	$\frac{s+a}{(s+a)^2+\omega^2}$

Table 2: Laplace transform pairs

Degrees	$0°$	$30°$	$45°$	$60°$	$90°$	$180°$	$270°$	$360°$
Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
Sine	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	0	-1	0
Cosine	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-1	0	1
Tangent	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undefined	0	undefined	0

Table 3: Angles

$$\operatorname{atan2}(y, x) = \begin{cases} \arctan\left(\frac{y}{x}\right) & \text{if } x > 0 \\ \arctan\left(\frac{y}{x}\right) + \pi & \text{if } x < 0 \text{ and } y \geq 0 \\ \arctan\left(\frac{y}{x}\right) - \pi & \text{if } x < 0 \text{ and } y < 0 \\ \frac{\pi}{2} & \text{if } x = 0 \text{ and } y > 0 \\ -\frac{\pi}{2} & \text{if } x = 0 \text{ and } y < 0 \\ \text{undefined} & \text{if } x = 0 \text{ and } y = 0 \end{cases}$$

Table 4: Polar coordinates

Name	Time domain function	z-domain function
Transform pair	$f[k]$	$F(z)$
Superposition	$\alpha_1 f_1[k] + \alpha_2 f_2[k]$	$\alpha_1 F_1(z) + \alpha_2 F_2(z)$
Time delay	$f[k-n]$	$z^{-n}\left(F(z) + \sum_{i=1}^{n} f[-i] z^{i}\right)$
Time advance	$f[k+n]$	$z^{n}\left(F(z) + \sum_{i=1}^{n} f[i] z^{-i}\right)$
Convolution	$f_1[k] * f_2[k]$	$F_1(z) F_2(z)$
	$= \sum_{i=0}^{k} f_1[k-i] f_2[i]$	
	$= \sum_{i=0}^{k} f_1[i] f_2[k-i]$	
Frequency scaling	$b^k f[k]$	$F\left(\frac{z}{b}\right)$
Complex modulation	$e^{ak} f[k]$	$F(ze^{-a})$
Power modulation	$a^k f[k]$	$F(a^{-1} z)$
Initial value theorem	$f[0]$	$\lim_{\lvert z\rvert \to \infty} F(z)$
Final value theorem	$\lim_{k \to \infty} f[k]$	$\lim_{z \to 1} (z-1) F(z)$

Table 5: Properties of the z-transform

Name	$f[k]$	$F(z)$
Impulse function	$\delta[k] = \left\{ \begin{array}{ll} 1, & k = 0 \\ 0 & k \neq 0 \end{array} \right\}$	1
Step function	$f[k] = 1$ for $k \geq 0$	$\frac{z}{z-1} = \frac{1}{1-z^{-1}}$
Power	b^k	$\frac{z}{z-b} = \frac{1}{1-bz^{-1}}$
Exponential	e^{ak}	$\frac{z}{z-e^{a}}$
Ramp sequence	k	$\frac{z}{(z-1)^2}$
Parabolic sequence	k^2	$\frac{z(z+1)}{(z-1)^3}$
Power ramp	$b^k k$	$\frac{bz}{(z-b)^2}$
General power	$\binom{k}{j-1} b^{k-(j-1)}$	$\frac{z}{(z-b)^j}$
Exponential ramp	$e^{ak} k$	$\frac{ze^{a}}{(z-e^{a})^2}$
Sine	$\sin[\omega k]$	$\frac{\sin(a) z}{z^2 - 2\cos(\omega) z + 1}$
Cosine	$\cos[\omega k]$	$\frac{z(z-\cos(a))}{z^2 - 2\cos(\omega) z + 1}$
Power sine	$b^k \sin[\omega k]$	$\frac{b \sin(\omega) z}{z^2 - 2b\cos(a) z + b^2}$
Power cosine	$b^k \cos[\omega k]$	$\frac{z(z - b\cos(\omega))}{z^2 - 2b\cos(a) z + b^2}$

Table 6: z-transform pairs

Continuous-time	Discrete-time
$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}u(t)$	$\mathbf{x}[k+1] = \mathbf{A}\mathbf{x}[k] + \mathbf{b}u[k]$
$y(t) = \mathbf{c}\mathbf{x}(t) + du(t)$	$y[k] = \mathbf{c}\mathbf{x}[k] + du[k]$
$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{b}u(\tau)d\tau$	$\mathbf{x}[k] = \mathbf{A}^k\mathbf{x}[0] + \sum_{j=1}^{k-1} \mathbf{A}^{k-j-1}\mathbf{b}u[j]$
$H(s) = \mathbf{c}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b} + d$	$H(z) = \mathbf{c}(z\mathbf{I} - \mathbf{A})^{-1}\mathbf{b} + d$
$h(t) = \mathbf{c}e^{\mathbf{A}t}\mathbf{b} + d\delta(t)$	$h[k] = \begin{cases} d & k = 0 \\ \mathbf{c}\mathbf{A}^{k-1}\mathbf{b} & k > 0 \end{cases}$
$h(t) = \mathfrak{L}^{-1}\{H(s)\} = \sum_i A_i e^{p_i t} + \sum_r \sum_{j=1}^{m_r} B_{r,j} \frac{t^{(j-1)}}{(j-1)!} e^{\tilde{p}_r t}$ (PFE)	$h[k] = \mathfrak{z}^{-1}\{H(z)\} = \sum_i A_i p_i^k + \sum_r \sum_{j=1}^{m_r} B_{r,j} \binom{k}{j-1} \tilde{p}_r^{k-(j-1)}$. (PFE)
$B_{r,j} = \frac{1}{(m_r-j)!} \left\{ \frac{d^{m_r-j}}{ds^{m_r-j}} (s - \tilde{p}_r)^{m_r} h(s) \right\} \big\rvert_{s=\tilde{p}_r}$	$B_{r,j} = \frac{1}{(m_r-j)!} \left(\frac{d^{m_r-j}}{dz^{m_r-j}} \frac{(z-\tilde{p}_r)^{m_r} H(z)}{z} \right) \big\rvert_{z=\tilde{p}_r}$
$(s\mathbf{I} - \mathbf{A})^{-1} \underset{\mathfrak{L}}{\overset{\mathfrak{L}^{-1}}{\rightleftarrows}} e^{\mathbf{A}t}$	$(z\mathbf{I} - \mathbf{A})^{-1} \underset{\mathfrak{z}}{\overset{\mathfrak{z}^{-1}}{\rightleftarrows}} \mathbf{A}^{k-1}$
If $\bar{\mathbf{x}}(t) = \mathbf{T}\mathbf{x}(t)$ then	If $\bar{\mathbf{x}}[k] = \mathbf{T}\mathbf{x}[k]$ then
$(\mathbf{A}, \mathbf{b}, \mathbf{c}, d) \xrightarrow{T} (\bar{\mathbf{A}}, \bar{\mathbf{b}}, \bar{\mathbf{c}}, \bar{d})$ where	$(\mathbf{A}, \mathbf{b}, \mathbf{c}, d) \xrightarrow{T} (\bar{\mathbf{A}}, \bar{\mathbf{b}}, \bar{\mathbf{c}}, \bar{d})$ where
$\bar{\mathbf{A}} = \mathbf{T}\mathbf{A}\mathbf{T}^{-1}$	$\bar{\mathbf{A}} = \mathbf{T}\mathbf{A}\mathbf{T}^{-1}$
$\bar{\mathbf{b}} = \mathbf{T}\mathbf{b}$	$\bar{\mathbf{b}} = \mathbf{T}\mathbf{b}$
$\bar{\mathbf{c}} = \mathbf{c}\mathbf{T}^{-1}$	$\bar{\mathbf{c}} = \mathbf{c}\mathbf{T}^{-1}$
$\bar{d} = d$	$\bar{d} = d$
$u(t) = \sin(\omega t)$,	$u[k] = \sin(\omega kT)$,
$y_s(t) = \lvert G(i\omega)\rvert \sin(\omega t + \angle G(i\omega))$	$y_s[k] = \lvert G(e^{i\omega T})\rvert \sin\left(\omega kT + \angle G(e^{i\omega T})\right)$

Table 7: Summary of continuous-time and discrete-time systems theory

	Observable canonical form	Controllable canonical form
$\mathbf{A}$	$\begin{pmatrix} -a_1 & 1 & 0 & 0 & \dots & 0 & 0 \\ -a_2 & 0 & 1 & 0 & \dots & 0 & 0 \\ -a_3 & 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -a_{n-2} & 0 & 0 & 0 & \dots & 1 & 0 \\ -a_{n-1} & 0 & 0 & 0 & \dots & 0 & 1 \\ -a_n & 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} -a_1 & -a_2 & \dots & -a_{n-1} & -a_n \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}$
$\mathbf{b}$	$\begin{pmatrix} b_1 - a_1 b_0 \\ b_2 - a_2 b_0 \\ \vdots \\ b_n - a_n b_0 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$
$\mathbf{c}$	$\begin{pmatrix} 1 & 0 & \dots & 0 \end{pmatrix}$	$\begin{pmatrix} b_1 - a_1 b_0 & b_2 - a_2 b_0 & \dots & b_n - a_n b_0 \end{pmatrix}$
d	b_0	b_0

Table 8: Canonical forms

$\mathbf{S}_1 = \mathbf{I}$	$a_1 = -\mathrm{tr}(\mathbf{A})$
$\mathbf{S}_2 = \mathbf{S}_1\mathbf{A} + a_1\mathbf{I}$	$a_2 = -\frac{1}{2}\mathrm{tr}(\mathbf{S}_2\mathbf{A})$
$\mathbf{S}_3 = \mathbf{S}_2\mathbf{A} + a_2\mathbf{I}$	$a_3 = -\frac{1}{3}\mathrm{tr}(\mathbf{S}_3\mathbf{A})$
$\vdots$	$\vdots$
$\mathbf{S}_n = \mathbf{S}_{n-1}\mathbf{A} + a_{n-1}\mathbf{I}$	$a_n = -\frac{1}{n}\mathrm{tr}(\mathbf{S}_n\mathbf{A})$
$(s\mathbf{I} - \mathbf{A})^{-1} = \frac{\mathbf{S}_1 s^{n-1} + \mathbf{S}_2 s^{n-2} + \ldots + \mathbf{S}_n}{s^n + a_1 s^{n-1} + \ldots + a_n}$	

Table 9: Fadeev algorithm

Parallel interconnection	Serial interconnection
$\mathbf{z}[k+1] = \begin{pmatrix} \mathbf{A}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_2 \end{pmatrix} \mathbf{z}[k] + \begin{pmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{pmatrix} r[k]$	$\mathbf{z}[k+1] = \begin{pmatrix} \mathbf{A}_1 & \mathbf{0} \\ \mathbf{b}_2\mathbf{c}_1 & \mathbf{A}_2 \end{pmatrix} \mathbf{z}[k] + \begin{pmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 d_1 \end{pmatrix} r[k]$
$y[k] = \begin{pmatrix} \mathbf{c}_1 & \mathbf{c}_2 \end{pmatrix} z[k] + (d_1 + d_2) r[k]$	$y[k] = \begin{pmatrix} d_2\mathbf{c}_1 & \mathbf{c}_2 \end{pmatrix} z[k] + (d_2 d_1) r[k]$

Table 10: State-space interconnections

$\mathbf{x}[k+1] = \mathbf{\Phi}\mathbf{x}[k] + \mathbf{\Gamma} u[k]$

$y[k] = \mathbf{c}\mathbf{x}[k] + du[k]$

$\mathbf{\Phi} = e^{\mathbf{A}T}$

$\mathbf{\Gamma} = \int_0^T e^{\mathbf{A}\gamma}\mathbf{b}\, d\gamma$

$\mathbf{M} = \begin{pmatrix} \mathbf{A} & \mathbf{b} \\ \mathbf{0} & 0 \end{pmatrix}$

$e^{\mathbf{M}T} = \begin{pmatrix} \mathbf{\Phi} & \mathbf{\Gamma} \\ \mathbf{0} & 1 \end{pmatrix}$

$$\begin{pmatrix} e^{\lambda_1 T} \\ e^{\lambda_2 T} \\ \vdots \\ e^{\lambda_n T} \end{pmatrix} = \begin{pmatrix} \lambda_1^{n-1} & \lambda_1^{n-2} & \dots & 1 \\ \lambda_2^{n-1} & \lambda_2^{n-2} & \dots & 1 \\ \vdots & \vdots & & \vdots \\ \lambda_n^{n-1} & \lambda_n^{n-2} & \dots & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}$$

In case of repeated eigenvalues λ_i, for each repetition differentiate lhs and rhs w.r.t. λ_i.

In case of complex values split it up, where you take the real part of both lhs and rhs and another one where you have both imaginary parts.

Table 11: Discrete-time equivalents of continuous-time systems

$\mathbf{W}_c = \begin{pmatrix} \mathbf{B} & \mathbf{AB} & \dots & \mathbf{A}^{n-1}\mathbf{B} \end{pmatrix}$

$$\bar{\mathbf{W}}_c^{-1} = \begin{pmatrix} 1 & a_1 & a_2 & \dots & a_{n-2} & a_{n-1} \\ 0 & 1 & a_1 & \ddots & a_{n-3} & a_{n-2} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & a_1 & a_2 \\ 0 & \dots & 0 & 0 & 1 & a_1 \\ 0 & \dots & 0 & 0 & 0 & 1 \end{pmatrix}$$

$\mathbf{T}^{-1} = \mathbf{W}_c \bar{\mathbf{W}}_c^{-1}$

$\mathbf{u}[k] = -\bar{L}\bar{\mathbf{x}}[k] = -\bar{L}\mathbf{T}\mathbf{x}[k] = -L\mathbf{x}[k]$

$\bar{L} = \begin{pmatrix} \bar{l}_1 & \bar{l}_2 & \dots & \bar{l}_n \end{pmatrix} = \begin{pmatrix} \tilde{a}_1 - a_1 & \tilde{a}_2 - a_2 & \dots & \tilde{a}_n - a_n \end{pmatrix}$

Table 12: Control