Mathematical Modeling. Intro to MM 01-02-21

Lecture I

Phenomenon - part of reality that is subject of study.
Has form of independance and internal cohes

e.g. Ball on field
Playing Chess.
Inverted pendulum on a cart.

Model - abstract rep. of reality.
- always Simplification of reality - not reality itself
- good model contains all structures & relationships that matter and ~~ignore~~ leave out all others of less relevance.

Math Model - relationships of phenomenon are represented by equations and ~~corresponding values~~ conditions on quantities.
- correspondance rule gives them meaning.

Modelling Cycle - Ensure all parties consent
Hardly ever right the first time
Sometimes client doesn't know what they want.

Problem Analysis
↓
Conceptual Modelling
↓
Math model class selection
↓
Conceptual validation
↓
Implementation ⟶ Verification
↑
System ident &calibration
↑
Model Validation
↑
Model Analysis
↑
Model Use

Running on the Unit Circle

$$(x(t), y(t)) = K(\cos(t), \sin(t))$$

$$x'(t) = -K\sin(t)$$

$$y'(t) = K\cos(t)$$

$$\dot{x} = -y$$
$$\dot{y} = x$$

$$\frac{dy}{dx} = \frac{-x}{y} \qquad y\,dy = -x\,dx$$

Integrating : $\frac{1}{2}y^2 = -\frac{1}{2}x^2 + c$

$$x^2 + y^2 = c$$

Lotka-Volterra models. - which move onto predator - prey.

Sensitivity Analysis - influence of initial conditions/ parameters
- reliability of our model.

Growth of a population.

- growth depends on pop. size.

$$x'(t) = \epsilon\, x(t)$$ growth factor.

- which kind of $x(t)$ solves this?

$$x(t) = e^{\epsilon t} x_0$$

- exponential growth
- unrestricted.

Exponential growth w/ limited food supply.

- Limited food supply
- decrease growth factor for large populations.

growth factor

$$x'(t) = [\epsilon - \sigma\, x(t)]\, x(t)$$

factor to reduce growth based on [pop.]

- if pop. is too large; negative growth (starvation).

How to measure the parameters?

- Exponential growth factor ϵ is easily observed directly in a small pop.

- σ can be measured in equillibrium stage, i.e at $x'(t) = 0$.

Lotka-Volterra models

$$x'(t) = (\epsilon - \sigma \cdot x(t))\, x(t)$$

$$x'(t) = (\epsilon - \sigma \cdot x(t)) \cdot x(t) = 0$$

$$\underbrace{x(t) = 0}_{\text{extinct}} \quad \text{or} \quad \epsilon - \sigma x(t) = 0$$

$$\Rightarrow \sigma = \frac{\epsilon}{x(t)} \text{ equillibrium}$$

Competing Species

- Two species fighting over shared limited food supply.
- Pop. of other species affects growth of the first.

two competing species

limit of growth of zebra by buffalo. pop. of waterbuffalo

$$\text{Zebras - } \underbrace{x'(t)}_{\text{pop. change}} = (\epsilon_1 - \sigma_1 x(t) - \alpha_1 y(t))\, x(t)$$

$$\text{Buffalo - } y'(t) = (\epsilon_2 - \sigma_2 y(t) - \alpha_2 x(t))\, y(t)$$

equillibria.

$$\begin{cases} x'(t) = 0 \\ y'(t) = 0 \end{cases}$$

Cases for equillibria

- $x(t) = 0 \wedge y(t) = 0$
- $x(t) = 0 \wedge \epsilon_2 - \sigma_2 y(t) - \alpha_2 \overset{0}{x(t)} = 0$

  $\Rightarrow y(t) = \dfrac{\epsilon_2}{\sigma_2}$

- $y(t) = 0 \wedge \epsilon_1 - \sigma_1 x(t) - \alpha_1 \overset{0}{y(t)} = 0$

  $\Rightarrow x(t) = \dfrac{\epsilon_1}{\sigma_1}$

- $\begin{cases} \epsilon_1 - \sigma_1 x(t) - \alpha_1 y(t) = 0 \\ \epsilon_2 - \sigma_2 y(t) - \alpha_2 x(t) = 0 \end{cases} \rightarrow \begin{cases} x(t) = \dfrac{\epsilon_1 - \alpha_1 y(t)}{\sigma_1} \\ y(t) = \dfrac{\epsilon_2 - \alpha_2 x(t)}{\sigma_2} \end{cases}$

Predator-Prey Models

- Prey pop. as before with possibly limited food.
- predator pop. that eats prey.
- predator pop. will starve if there are no prey.

Prey

too many zebras they starve

Zebras $\quad x'(t) = (\epsilon_1 - \sigma_1 (x) t - \alpha_1 y(t))\, x(t)$

$= \epsilon_1 x(t) - \sigma_1 (x^2(t)) - \alpha_1 x(t)\, y(t)$

lions affect zebra pop.

Predators.

Lions. $\quad y'(t) = (-\epsilon_2 + \alpha_2 x(t) - \sigma_2 y(t))\, y(t)$

decreasing pop if no zebras. more zebras = lions can eat. too many lion will starve lions.

$= -\epsilon_2 y(t) + \alpha_2 x(t)\, y(t) - \sigma_2 (y^2(t))$.

Jacobian

derivative w.r.t. x $\qquad$ w.r.t. y.

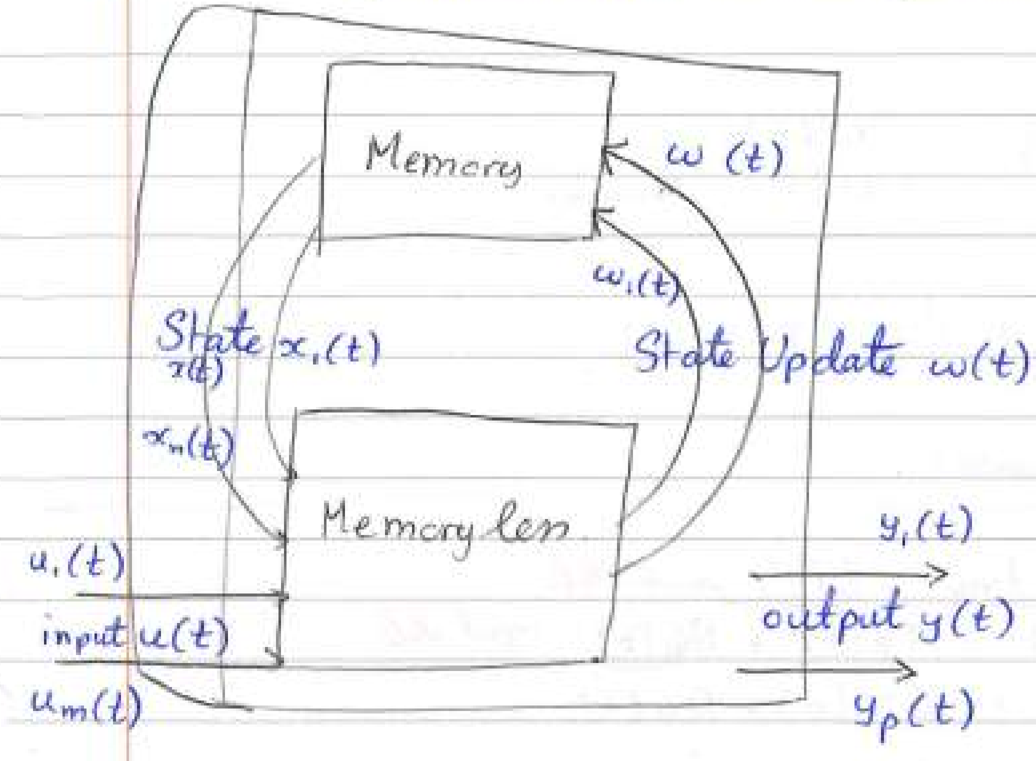
$$\begin{array}{l} x'(t) \to \\ y'(t) \to \end{array} J = \begin{bmatrix} \epsilon_1 - 2\sigma_1 x - \alpha_1 y & -\alpha_1 x \\ \alpha_2 y(t) & \epsilon_2 - 2\sigma_2 y + \alpha_2 x \end{bmatrix}$$

at $(0,0)$ linearise this as.

$$\begin{bmatrix} \epsilon_1 & 0 \\ 0 & \epsilon_2 \end{bmatrix} \qquad \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} \epsilon_1 & 0 \\ 0 & \epsilon_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Mathematical Modeling Lecture 2.

08/02/21



<u>output</u>

it is also dependant on time!

$$y_k(t) = g(x_1(t), x_2(t), \dots, x_n(t), u_1(t), \dots, u_n(t), t)$$

$$x(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix} \qquad y(t) = \begin{pmatrix} y_1(t) \\ \vdots \\ y_p(t) \end{pmatrix}$$

output

$$y(t) = g\underbrace{(x(t), u(t), t)}_{\text{vectorised}} \qquad u(t) = \begin{pmatrix} u_1(t) \\ \vdots \\ u_m(t) \end{pmatrix}$$

state update

This function is combined $x'(t)$ & $y'(t)$ of predator prey. so the growth of both combined.

state is for example the pop. of both the lions and zebras.

combined pops

$$\omega(t) = f(x(t), u(t), t)$$

discrete-time $\omega(t) = x[t+1]$ — square bracket denote discrete.

continuous-time $\omega(t) = x'(t)$.

if g and f are time-invariant & and linear.

$$y_k(t) = (c_1 x_1(t) + c_2 x_2(t) + \dots + c_n x_n(t) + d_1 u_1(t) + \dots + d_m u_m(t))$$

$$\underset{p\times 1}{y(t)} = \overset{p\times n}{C}\underset{n\times 1}{x(t)} + \overset{p\times m}{D}\underset{m\times 1}{u(t)} \quad \therefore \text{MIMO} \quad \text{LTI}$$

multi input-multi output   Linear Time Invariant.

$w(t)$ in continuous time.

$$\underset{n\times m}{x'(t)} = \underset{n\times n}{A}\, x(t) + \underset{n\times m}{B}\, u(t)$$

$$w(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \begin{matrix} \text{- Prey} \\ \text{- Predator.} \end{matrix}$$

state

$$y(t) = (1 \ \ 1)\ w(t)$$

output.

(toy example that counts total wildlife).

<u>discrete-time LTI</u>

$$x[k+1] = A\,x[k] + B\,u[k]$$

previous state — input state

$$y[k] = C x[k] + D u[k]$$

<u>CT</u> (continuous time).

$$x'(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

<u>Species Metamorphoses</u>

rate of convergence.

$$(A) \xrightarrow{k_1} (B) \xrightarrow{k_2} (C)$$

state $x(t)$

$x_1(t) =$ pop size A

$x_2(t) =$ pop. size B

$x_3(t) =$ pop size C

$$\dot{x}_1(t) = -k_1 x_1(t)$$
$$\dot{x}_2(t) = k_1 x_1(t) - k_2 x_2(t)$$
$$\dot{x}_3(t) = k_2 x_2(t)$$

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}$$

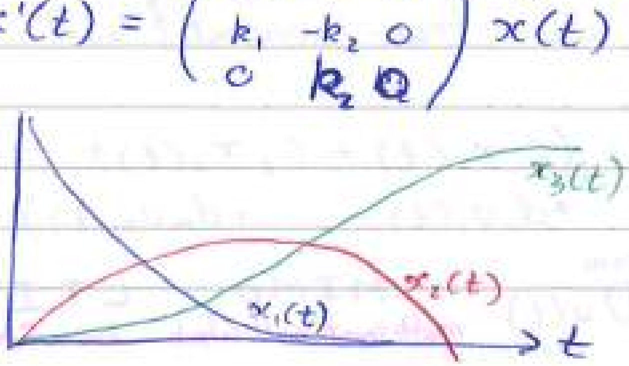
$$x'(t) = \overbrace{\begin{pmatrix} -k_1 & 0 & 0 \\ k_1 & -k_2 & 0 \\ 0 & k_2 & 0 \end{pmatrix}}^{A} \underbrace{x(t)}_{\begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}}$$

Total pop $= x_1 + x_2 + x_3$
$(1 \ \ 1 \ \ 1)\, x(t)$

$$\frac{d}{dt}\left(1^T x(t)\right) = 1^T x'(t)$$
$$= (1^T A)\, x(t) = 0$$

column sum of A which $= 0$.

Note: we have conservation of population.



$$y(t) = (1 \ 1 \ 1)\, x(t)$$

B one input.

$$x'(t) = \begin{pmatrix} -k_1 & 0 & 0 \\ k_1 & -k_2 & 0 \\ 0 & k_2 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} u(t)$$

same as A

$$x_1'(t) = -k_1 x_1(t) + \mu(t)$$ one from previous B. (birth of the eggs)

$$x_2'(t) = k_1 x_1(t) - k_2 x_2(t)$$

$$x_3'(t) = k_2 x_2(t) - d x_3(t)$$ death of the butterflies.

Markov Chains.

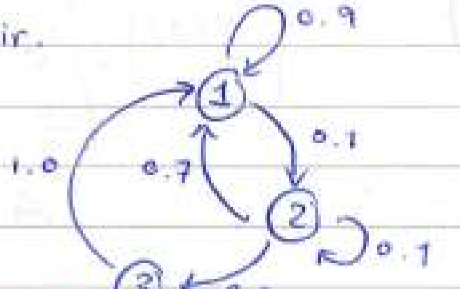
$Z(t) \in [1, \dots, n]$ random sequence

$$\text{Prob}\,(Z[t+1] = i \mid Z[t] = j) = \underline{P}_{ij}$$

$$p[t] = \begin{bmatrix} \text{Prob}(Z[t]=1) \\ \vdots \\ \text{Prob}(Z[t]=n) \end{bmatrix}$$

Markov chain windows computer state

1. working
2. down
3. in repair.



$$\begin{array}{l c} & \begin{array}{ccc} \text{if in 1} & \text{in 2} & \text{in 3.} \end{array} \\ \begin{array}{l} \text{P(going to 1)} \\ \text{p(going to 2)} \\ \text{p(going to 3)} \end{array} & \underline{P} = \begin{bmatrix} 0.9 & 0.7 & 1.0 \\ 0.1 & 0.1 & 0 \\ 0 & 0.2 & 0 \end{bmatrix} \end{array}$$

$$x[0] = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$ (working comp.)

$$x[k+1] = \underline{P}\, x[k]$$

steady state
$$x[k+1] = x[k]$$

$$\overset{\lambda}{\overset{\shortparallel}{1}}\, x = \underline{P}\, x \qquad P x[k] = x[k]$$

$$(\underline{P} - \overset{1}{\overset{\shortparallel}{\lambda}} I)\, v = 0$$ eigenvector.

$$\begin{pmatrix} -0.1 & 0.7 & 1 \\ 0.1 & -0.9 & 0 \\ 0 & 0.2 & -1 \end{pmatrix} v = 0 \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0.2 & -1 & 0 \\ 0.1 & -0.9 & 0 & 0 \end{array}\right)$$

but this is linearly dependent

get rid of top row then entries sum to 1.

but the prob must sum up to 1.

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e.x. traffic lights.
red or green
if red, it will stay red w.p. 2/3 and will turn green w.p. 1/3
if green, stay green w.p. 1/2, turn red w.p. 1/2.

start red. $x[0] = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$\begin{array}{r c} & \begin{array}{cc} \text{red} & \text{green} \end{array} \\ \begin{array}{r} \text{red} \to \\ \text{green} \to \end{array} & \underline{P} = \begin{pmatrix} 2/3 & 1/2 \\ 1/3 & 1/2 \end{pmatrix} \end{array}$$

$$x[1] = \underline{P}\, x[0] = \begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix}$$

$$x[2] = \underline{P}\, x[1] = \underline{P}^2 x[0] = \underline{P}\,\underline{P}\, x[0]$$

$$= \begin{pmatrix} (2/3)^2 + 1/6 \\ 2/9 + 1/6 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0.2 & -1 & 0 \\ 0.1 & -0.9 & 0 & 0 \end{array}\right) \quad \begin{array}{l} \times 5 \\ -5\,① \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 6 & 1 \\ 0 & 1 & -5 & 0 \\ 0 & -1 & -0.1 & -0.1 \end{array}\right) \quad -0.1\,①$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 6 & 1 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & -5.1 & -0.1 \end{array}\right) \quad +②$$

$$\frac{-100}{51} \times \underset{\frac{-51}{10}}{=} \quad \underset{\frac{-1}{10}}{=} \times \frac{-100}{51}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1-\frac{6}{51} \\ 0 & 1 & 0 & \frac{5}{51} \\ 0 & 0 & 1 & \frac{1}{51} \end{array}\right) \quad \begin{array}{l} -6\,③ \\ +5\,③ \end{array}$$

$$v = \begin{pmatrix} 45/51 \\ 5/51 \\ 1/51 \end{pmatrix}$$

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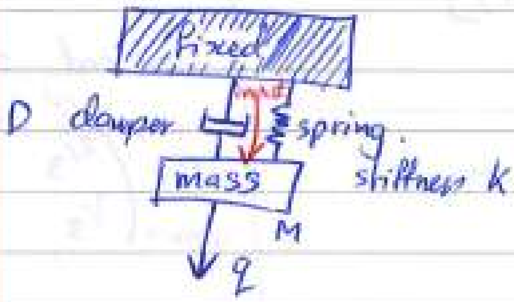
mass-damper-spring.



$$M\ddot{q} + D\dot{q} + kq = 0$$

$$\ddot{q} = \frac{-D}{M}\dot{q} - \frac{k}{M}q$$

$$x(t) = \begin{pmatrix} q(t) \\ \dot{q}(t) \end{pmatrix}$$

$$\dot{x}(t) = \begin{pmatrix} & \end{pmatrix} x(t)$$

$$\begin{pmatrix} \dot{q}(t) \\ \ddot{q}(t) \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ \frac{-k}{M} & \frac{-D}{M} \end{pmatrix}}_{A} \begin{pmatrix} q(t) \\ \dot{q}(t) \end{pmatrix}$$

input

$$F = ma$$

$$\Rightarrow a = \frac{F}{m}$$

$$x'(t) = A \quad x(t) + B \quad u(t)$$

$$\begin{pmatrix} \dot{q}(t) \\ \ddot{q}(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{-k}{M} & \frac{-D}{M} \end{pmatrix} \begin{pmatrix} q(t) \\ \dot{q}(t) \end{pmatrix} + \begin{pmatrix} 0 \\ 1/M \end{pmatrix} u(t)$$

only output. $- \; y(t) = (1 \;\; 0)\, x(t) + \emptyset\, u(t)$

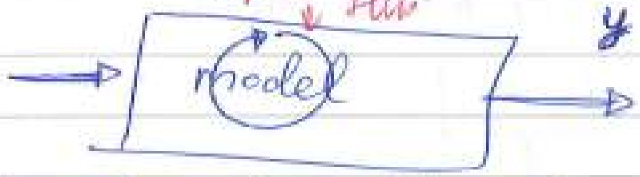
$$\overset{\shortparallel}{q(t)} \quad \begin{pmatrix} q(t) \\ \dot{q}(t) \end{pmatrix}$$

Mathematical Modelling Lecture 3

Continuous-time transfer, Laplace transform & impulse response

now we want to generalize

inputs u — so far looked at this — outputs y



Ex. of second order linear diff. equation

$$* \Rightarrow \quad y''(t) + 3y'(t) - y(t) = 4u''(t) + u'(t) + 3u$$

Laplace Transformation

$$f(t) \xrightarrow{\mathcal{L}} F(s) := \int_0^\infty f(t) e^{-st}\, dt$$

$c_1 f_1(t) + c_2 f_2(t)$, can split up integral $= c_1 F_1(s) + c_2 F_2(s)$ (linear transformation)

lower case for time: time func.

upper case for transformed.

we consider $f(t)$ which have $f(t) = 0$ for $t < 0$, thus we start at 0.

s complex

- this is one sided $\mathcal{L}$
- can generalise to two-sided $\mathcal{L}$ with $(-\infty)$.

$s = i\omega \quad (\omega \in \mathbb{R})$

s relates to frequencies (more complex def. will follow)

$$f(t) = e^{2t}$$

$$\boxed{F(s)} = \int_0^\infty e^{2t} \cdot e^{-st}\, dt = \int_0^\infty e^{(2-s)t}\, dt = \frac{1}{2-s} e^{(2-s)t} \Big|_0^\infty$$

we consider Re

$$= 0 - \frac{1}{2-s} \times e^{(2-s)0} = 0 - \frac{1}{2-s} = \boxed{\frac{1}{s-2}}$$

$u(t) \xrightarrow{\mathcal{L}} U(s)$

$y(t) \xrightarrow{\mathcal{L}} Y(s)$

$f(t) \xrightarrow{\mathcal{L}} F(s)$

$f'(t) \xrightarrow{\mathcal{L}} sF(s) - \underbrace{f(0)}_{=0} = sF(s)$

we look at condition where initially zero

$\mathcal{L}$ (Laplace transformation).

$f''(t) \xrightarrow{\mathcal{L}} s^2 F(s)$

$$* \quad s^2 Y(s) + 3sY(s) - Y(s) = 4s^2 U(s) + sU(s) + 3U(s)$$

$$(s^2 + 3s - 1)\, Y(s) = (4s^2 + s + 3)\, U(s)$$

$$\boxed{Y(s) = \frac{4s^2+s+3}{s^2+3s-1} \cdot U(s)}$$

transform of output depends on → transform of input multiplied by some value/constant

$H(s)$ transfer func.

$$\boxed{Y(s) = H(s) \cdot U(s)}$$

output — system — input.

Ex. of '$\mathcal{L}$'

$$f(t) = x'(t) = 3x(t) - 5e^{-2t} + 1 \qquad x(0) = 6.$$

$sF(s) - f(0)$ $\quad cf(t) \to cF(s)$ $\quad$ in table $\quad$ in table

$$sX(s) - 6 = 3X(s) - 5 \cdot \frac{1}{s+2} + \frac{1}{s}$$

$$(s-3)X(s) = -\frac{5}{s+2} + \frac{1}{s} + 6 \qquad e^{at} \,|\, \frac{1}{s-a}$$

$$X(s) = \frac{-5}{(s+2)(s-3)} + \frac{1}{s(s-3)} + \frac{6}{(s-3)}$$

$\Downarrow \mathcal{L}^{-1}$

$$6e^{3t}$$

$$\frac{-5}{(s+2)(s-3)} = \frac{1}{(s+2)} - \frac{1}{(s-3)}$$ — we already have these in the $\frac{1}{s-a}$ form

$$\frac{1}{s(s-3)} = \frac{-1/3}{s} + \frac{1/3}{s-3}$$

$$e^{-2t} - e^{3t} - \frac{1}{3} + \frac{1}{3}e^{3t}$$

$$\boxed{x(t) = e^{-2t} + \frac{16}{3}e^{3t} - \frac{1}{3}}$$

check $\quad x(0) = 1 + \frac{16}{3} - \frac{1}{3} = 6$ //

Verify. LHS $= x'(t) = -2e^{-2t} + 3(x(0) - \frac{2}{3})e^{3t} = -2e^{-2t} + (3x(0)-2)e^{3t}$

RHS $= 3(x(t)) - 5e^{-2t} + 1 = 3e^{-2t} + (3x(0)-2)e^{3t} - 1 - 5e^{-2t} + 1$

$= -2e^{2t} + (3x(0)-2)e^{3t}$

equal

no a_0 as $a_0 = 1$ by convention.

$$y^{(n)}(t) + a_1 y^{(n-1)}(t) + \dots + a_n y(t) =$$

$$= b_0 u^{(n)}(t) + b_1 u^{(n-1)}(t) + \dots + b_n u(t)$$

n^{th} order diff. equ.

- solving high order linear D.E. may be difficult but w/ '$\mathcal{L}$' just multiply by $H(s)$
- the output obtained from input by the "action" of the system, can use to tell which outputs affect which inputs, and what from the dynamics of the system.

$$\boxed{H(s) = \frac{b_0 s^n + b_1 s^{n-1} + \dots + b_n}{s^n + a_1 s^{n-1} + \dots + a_n}}$$

linear transformation of input.

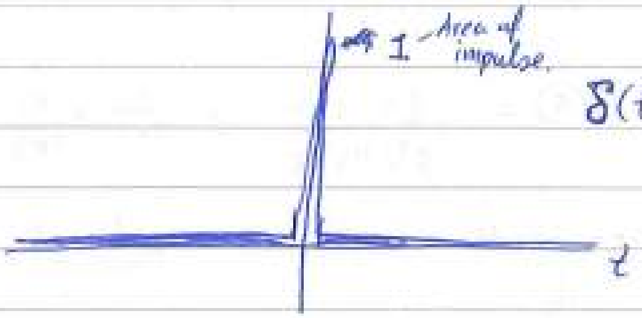
$$Y(s) = H(s)\, U(s)$$

if $U(s) = 1$ then $Y(s) = H(s)$

$\Uparrow$ $\qquad y(t) = h(t)$

$u(t) = ?$

impulse function:



$\delta(t)$ Dirac

when this func gets higher the duration gets relatively shorter and at ∞ it ends up as $\delta(t)$.

Area under the func. is 1 all the time. (can have multiples of $\delta(t)$ "delta func."

$$\Delta(s) \int_0^{\infty} \delta(t) e^{-st} dt = 1$$

$$\int_0^{\infty} \delta(t) f(t) dt = f(0) \quad \text{and } e^0 = 1$$

So back to

$$U(s) = 1 \qquad Y(s) = H(s)$$

$$\Uparrow \qquad y(t) = h(t)$$

$u(t) = \delta(t)$ — impulse (input)

impulse response (output).

Laplace transform of impulse response = transfer func.

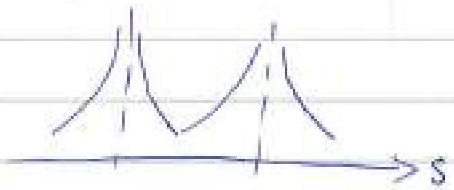
$$h(t) \xrightarrow{\mathcal{L}} H(s)$$

going from $H(s)$ (rational, degree n)
to a sum of easier rational functions all of degree ≤ 1.

partial fraction expansion (PFE)

$$H(s) = \frac{2s+3}{s^2+4s+3} = \frac{c_1}{s+1} + \frac{c_2}{s+3}$$

zeros of the denominator = poles of $H(s)$



$$s^2+4s+3 = (s+1)(s+3)$$

$$\frac{c_1(s+3)+c_2(s+1)}{(s+1)(s+3)} = \frac{(c_1+c_2)s + 3c_1+c_2}{\cdots}$$

Ex.

$$H(s) = \frac{26}{s^3+s^2-7s-15} \qquad h(t) = ?$$

step 1: poles of $H(s)$: $s^3+s^2-7s-15=0$

$3?$ $27+9-21-15=0$. factor $s-3$.

$$s^3+s^2-7s-15 = s-3\underbrace{(s^2+4s+5)}_{=0} \quad : s = -2 \pm i$$

$$H(s) = \frac{26}{(s-3)(s+2-i)(s+2+i)}$$

$$H(s) = \frac{26}{(s-3)(s+2-i)(s+2+i)}$$

$$= \underset{0}{\overset{\parallel}{A_0}} + \frac{A_1}{s-3} + \frac{A_2}{s+2-i} + \frac{A_3}{s+2+i}$$

$$H(s)(s-3) = \frac{26}{s^2+4s+5} = A_1 + \frac{A_2(\cancel{s-3})}{(s+2-i)} + \frac{A_3(\cancel{s-3})}{(s+2+i)}$$

plug in $s=3$

$$\frac{26}{9+12+5} = A_1 = 1$$

Ex. worked out in notes. $A_2 = \frac{-1+5i}{2}$ $A_3 = \frac{-1-5i}{2}$

$A_2 = \left.\frac{26}{(s-3)(s+2+i)}\right|_{s=-2+i} = \frac{26}{(-5+i)\cdot 2i} = \frac{26}{-2-10i} = \frac{-1+5i}{2}$, $A_3 = A_2^* = \frac{-1-5i}{2}$

Partial Fraction Expansion Example.

$H(s) = \frac{2s^3-3s^2+7s+4}{s^3+6s^2+11s+6}$, find $h(t)$

Step 1: find the poles of $H(s)$, need to solve $s^3+6s^2+11s+6=0$, gives $-1, -2$ & -3.
writing $s^3+6s^2+11s+6 = (s+1)(s+2)(s+3)$ we get.

$$\frac{2s^3-3s^2+7s+4}{(s+1)(s+2)(s+3)} = A_0 + \frac{A_1}{(s+1)} + \frac{A_2}{(s+2)} + \frac{A_3}{(s+3)}$$

$$\frac{2s^3-3s^2+7s+4}{(s+2)(s+3)} = A_0(s+1) + A_1 + \frac{A_2(s+1)}{(s+2)} + \frac{A_3(s+1)}{(s+3)}$$

sub $s=-1$

$\frac{-2-3-7+4}{2} = A_1 \Rightarrow A_1 = -4$

sub $s=-2$

$\frac{-16-12-14+4}{-1\cdot 1} = A_2 \Rightarrow A_2 = 38$

sub $s=-3$

$\frac{-54-27-21+4}{-2\cdot -1} = A_3 \Rightarrow A_3 = -49$.

As $s \to \infty$, $\frac{2s^3}{s^3 \ldots} \to 2$. $A_0 = 2$. As $s \to \infty$, equation tend to 2.

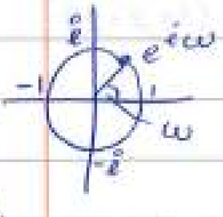
$h(t) = 2\delta(t) - 4e^{-t} + 38e^{-2t} - 49e^{-3t}$

$$H(s) = \frac{1}{s-3} + \frac{-1+5i}{2} \cdot \frac{1}{s+2-i} + \frac{-1-5i}{2} \cdot \frac{1}{s+2+i}$$

$$\mathcal{L}^{-1} \downarrow$$

$$h(t) = e^{3t} + \frac{-1+5i}{2} \cdot e^{(-2+i)t} + \frac{-1-5i}{2} \cdot e^{(-2-i)t}$$

$$e^{(-2+i)t} = e^{-2t} \cdot e^{it} = e^{-2t}(\cos(t) + i\sin(t))$$



$$e^{(-2-i)t} = e^{-2t} \cdot e^{-it} = e^{-2t}(\cos(t) - i\sin(t))$$

Ch4. Ex 4.
HW.

$$\cancel{h(t) = e^{3t} \; \frac{-1+5i}{2}\left(e^{-2t}(\cos(t)+i\sin(t))\right) \; \frac{-1-5i}{2}\left(e^{-2t}(\cos(t)-i\sin(t))\right.}$$

$$\cancel{= \frac{e^{3t}}{2}\left(-1+5i\left(e^{-2t}(\cos(t)+i\sin(t))\right) - 1-5i \, e^{-2t}(\cos(t)+i\sin(t))\right)}$$

$$\cancel{= \frac{e^{3t}}{2}\left(\left(-e^{-2t} + 5i\,e^{-2t}\right)(\cos(t)+i\sin(t)) - \left(e^{-2t} - 5i\,e^{-2t}\right)(\cos(t) - i\sin(t))\right)}$$

$$\cancel{= \frac{e^{3t}}{2}\left(\cos(t) + i\sin(t)\right.}$$

Mathematical Modelling Lecture 4

Continuous-time transfer functions with repeated roots and the z-domain.

Exe. Nr.7.

$$H(s) = -\frac{s^2-12s+11}{s^3+s+10} \qquad h(t)$$

impulse

impulse response

input $\longrightarrow$ $\boxed{H(s)}$ $\longrightarrow$ output

$Y(s) = H(s)\,U(s)$

$u(t) = \delta(t) \qquad U(s) = \Delta(s) = 1$

impulse

$$H(s) = -\frac{s^2-12s+11}{s^3+s+10}$$

$$= -\frac{s^2-12s+11}{(s+2)(s-(1+2i))(s-(1-2i))}$$

$$= A_0 + \boxed{\frac{A_1}{(s+2)} + \frac{A_2}{(s-(1+2i))} + \frac{A_3}{(s-(1-2i))}}$$

$s^3+s+10 = 0$

$s=-2$ solution

$s+2$ is a factor

$s^3+s+10 = (s+2)(s^2-2s+5)$

$s^2-2s+5 = 0 \qquad s = 1 \pm 2i$

$$\frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{+2\pm\sqrt{4-4\cdot 5}}{2} = \frac{2\pm\sqrt{-16}}{2} = \frac{2\pm 4\sqrt{-1}}{2} = 1\pm 2i$$

$s-(1+2i)=0$

$s-(1-2i)=0$

$$A_1 = H(s)(s+2) = A_1 + \frac{A_2(s+2)}{s-(1+2i)} + \frac{A_3(s+2)}{s-(1-2i)}\Bigg|_{s=-2}$$

$$A_1 = -\frac{s^2+12s-11}{s^2-2s+5}\Bigg|_{s=-2} = \frac{-4-24-11}{4+4+5} = \frac{-39}{13} = -3$$

$$A_2 = H(s)(s-(1+2i))\Big|_{s=1+2i}$$

$$= -\frac{s^2+12s-11}{(s+2)(s-(1-2i))}\Bigg|_{s=1+2i} = \frac{(1+2i)^2+12(1+2i)-11}{(1+2i+2)(1+2i-1+2i)}$$

$A_2 = 1-i$

$$= \frac{(1+4i-4)+12+24i-11}{-8+12i}$$

$A_3 = 1+i$

$$= \frac{4+20i}{-8+12i} = \frac{1+5i}{-2+3i}\cdot\frac{-2-3i}{-2-3i}$$

$$= \frac{-2-10i-3i+15}{4+9} = \frac{13-13i}{13} = 1-i$$

$$H(s) = \frac{-3}{s+2} + \frac{1-i}{s-(1+2i)} + \frac{1+i}{s-(1-2i)}$$

$$h(t) = -3e^{-2t} + (1-i)\cdot e^{(1+2i)t} + (1+i)\cdot e^{(1-2i)t}$$

$e^{i\phi} = \cos(\phi) + i\sin(\phi)$

$$(1-i)e^{t}e^{2it}$$

$$(1-i)e^{t}(\cos(2t) + i\sin(2t))$$

$$e^{t}(\cos(2t) - \cancel{i\cos(2t)} + \cancel{i\sin(2t)} + \sin(2t))$$

$$* \to e^{t}(\cos(2t) + \cancel{i\cos(2t)} - \cancel{i\sin(2t)} + \sin(2t)) \quad +$$

$$\Rightarrow 2e^{t}(\cos(2t) + \sin(2t))$$

$$h(t) = -3e^{-2t} + 2e^{t}(\cos(2t) + \sin(2t))$$

$$H(s) = \frac{1}{(s+1)(s+2)^2}$$

$$H(s) = \frac{A_1}{(s+1)} + \frac{A_2}{(s+2)} + \frac{A_3}{(s+2)^2}$$

damped polynomial in short.

$$h(t) = A_1 e^{-t} + A_2 e^{-2t} + A_3 t e^{-2t} \qquad t^k e^{at} \to \frac{k!}{(s-a)^{k+1}}$$

Suppose there is a pole at $\boxed{s=q}$ with multiplicity m.

$$H(s) = \frac{\text{-----}}{(s-q)^m(\text{----})}$$

include: $\frac{B_1}{s-q} + \frac{B_2}{(s-q)^2} + \frac{B_3}{(s-q)^3} + \cdots + \frac{B_m}{(s-q)^m}$ + other terms.

$H(s)\cdot(s-q)^m$

$$B_1(s-q)^{m-1} + B_2(s-q)^{m-2} + \cdots\cdots + B_{m-1}(s-q) + B_m$$

+ other terms $\cdot (s-q)^m$

set $s=q \Rightarrow B_m$

$$H(s)\cdot(s-q)^m \Big|_{s=q} = B_m$$

diff. $H(s)(s-q)^m$: $\qquad (s-q)' = 1$

$$(m-1)B_1\cancel{(s-q)}^{m-2} + (m-2)\cancel{B_2}(s-q)^{m-3} + \dots + B_{m-1} \cdot \overset{'}{1} + 0 + \cancel{(s-q)^{m-1}} \text{ other terms}$$

$$\Big|_{s=q} \Rightarrow B_{m-1}$$

$$\frac{d^k}{ds^k} H(s)(s-q)^m \Big|_{s=q} = k!\ B_{m-k}$$

BIBO stability

bounded input bounded output.

$$u(t) \longrightarrow \boxed{H(s)} \longrightarrow y(t)$$

$$Y(s) = H(s)\,U(s)$$

$$y(t) = \int_0^t h(\tau)u(t-\tau)\,d\tau$$

convolution integral between $u(t)$ $h(t)$ $= \int_0^t u(\tau)h(t-\tau)\,d\tau$

Suppose $\int_0^\infty |h(t)|\,dt = M < \infty$

$$|y(t)| = \left| \int_0^t h(\tau)u(t-\tau)\,d\tau \right| \le \int_0^t |h(\tau)|\underbrace{|u(t-\tau)|}_{\text{bounded by } B}\,d\tau$$

$$\le \int_0^t |h(\tau)|\,dt \cdot B$$

$\int_0^\infty |h(\tau)|d\tau = M$

so $\int_0^t |h(\tau)|d\tau \le M$ $\quad \le M \cdot B$

$H(s) = \frac{1}{s+5} + \frac{1}{s+2} \longrightarrow h(t) = e^{-5t} + e^{-2t}$

$$\int_0^\infty |h(t)|\,dt < \infty$$

$$\int_0^\infty e^{-5t}dt = -\frac{1}{5}e^{-5t}\Big|_0^\infty = \left[0 + \frac{1}{5}e^0\right] = \frac{1}{5}$$

BIBO stable.

$$\int_0^\infty e^{-2t}dt = -\frac{1}{2}e^{-2t}\Big|_0^\infty = \left[0 + \frac{1}{2}\right] = \frac{1}{2}$$

BIBO stable

$\frac{1}{s-5} \rightarrow e^{5t} \xrightarrow[t\to\infty]{} \infty$

not BIBO stable.

$3e^{(-2+i)t} + 3e^{(-2-i)t}$

$3e^{-2t} \cdot e^{it} + 3e^{-2t} \cdot e^{-it}$ absolute values of these, as they are purely imaginary, will be 1.

You will always converge if $\underbrace{e^{\text{-ive real}} \text{ or } e^{\text{-ive real part}}}_{\text{this is the only part that matters.}} {}^{+\text{ whatever } i}$

BIBO stability for our class of systems

↳ it requires that all poles have a -ive real part.

equivalent!

Asymptotic stability

if the output goes to zero if we apply a zero input (start from any arbitrary initial position)

if $h(t) \to 0$ for $t \to \infty$: asymptotic stability
$\to 0$ not so

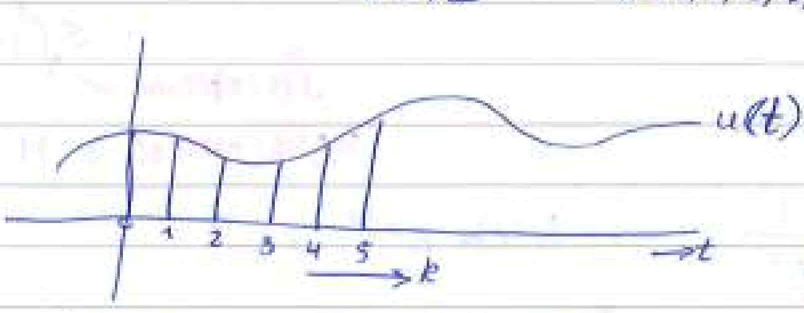
⇓

CONTINUOUS TIME

---

continuous-time: $u(t)$ $y(t)$, $x(t)$
$t \in \mathbb{R}$ $[0, \infty)$

discrete-time: time (index) k
$k \in \mathbb{Z}$ $k = 0, 1, 2, 3, \ldots\ldots$



Δt : sampling time = interval between consecutive samples

$\{u[0], u[1], u[2], \ldots\ldots \}$

$\boxed{t = k \cdot \Delta t}$ $u[2] = u(2 \cdot \Delta t)$

Difference equations instead of differential equations.

$$x'(t) = a\,x(t) \qquad x(0) = x_0$$

$$x(t) = e^{at} x_0$$

$\Delta t = 1$ $\qquad x[k] = x(k) = e^{ak} x_0$

$$e^{0} x_0,\ e^{a} x_0,\ e^{2a} x_0,\ e^{3a} x_0, \ldots\ldots$$

$$\overset{\shortparallel}{x_0},\ e^{a} x_0,\ (e^{a})^2 x_0,\ (e^{a})^3 x_0, \ldots\ldots$$

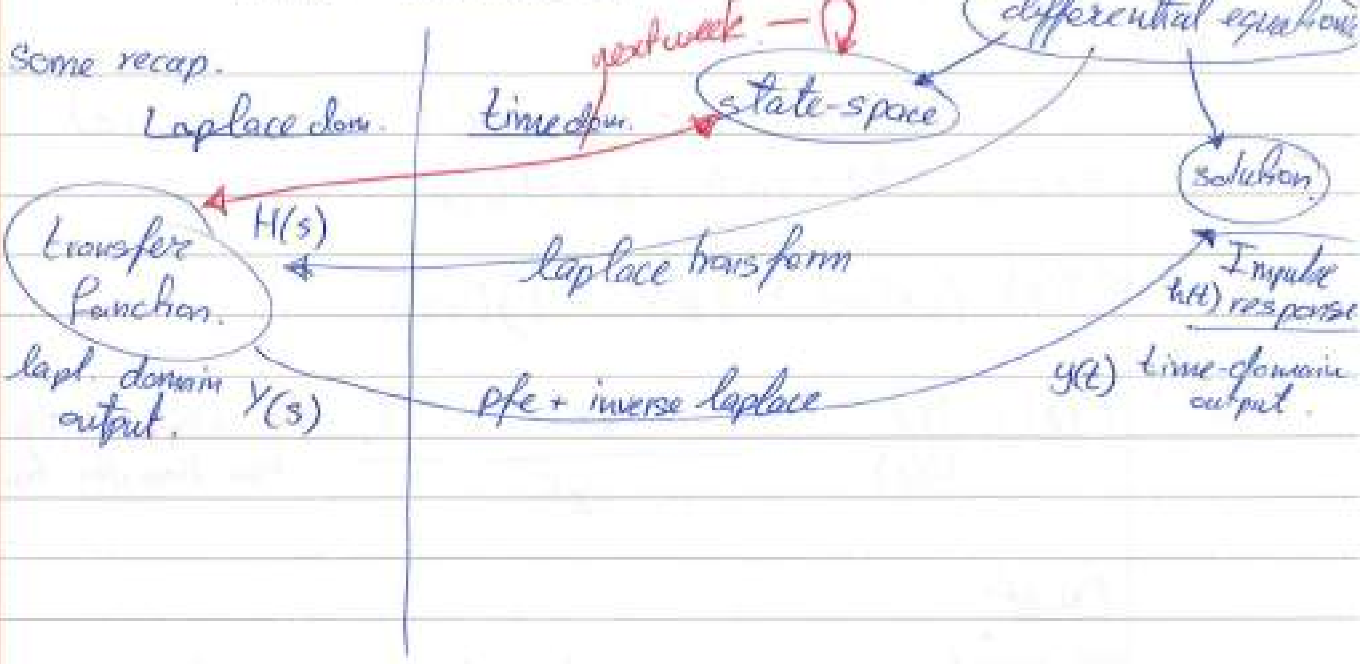
$$x[0],\ x[1]\ \ x[2]\ \ x[3]$$

$$x[k+1] = e^{a} \cdot x[k] \qquad x[0] = x_0$$

difference equation

MM- Lecture 5.

Some recap.



Today.

<u>Discrete-Time</u> (of all the stuff above)

sampling interval $T = 1$ day. $= 24 \cdot 60 \cdot 60s = 86{,}400s$

sampling frequency $f_s = \frac{1}{T} = 0.0000115740 7\ Hz$

7 day average can be written as a difference equation

$$y[k] = \frac{1}{7} u[k] + \frac{1}{7} u[k-1] + \frac{1}{7} u[k-2] + \dots + \frac{1}{7} u[k-6]$$

k is time index, corresponding to time $t_0 + kT$

[ ] to indicate discrete.

<u>~~Bo~~ Z-transform</u>

differential equation — derivatives — Laplace.

difference equation — time delays — Z-transform

Z-transform of sequence $\{f[k]\}$ — assume $f[k]=0 \quad \forall k<0$

$$F(z) = \mathcal{Z}\{f[k]\} := \sum_{k=0}^{\infty} f[k] z^{-k}$$

z is a complex var.

to note similarities between discrete & continuous.

$$F(s) = \mathcal{L}\{f(t)\} := \int_0^{\infty} f(t) e^{-st} dt$$

complex exponential

Ex. $y[k] = \frac{1}{7}u[k] + \frac{1}{7}u[k-1] + \frac{1}{7}u[k-2] + \dots + \frac{1}{7}u[k-6]$

$\mathcal{Z}$

$Y(z) = \frac{1}{7}U(z) + \frac{1}{7}z^{-1}U(z) + \frac{1}{7}z^{-2}U(z) + \dots\dots + \frac{1}{7}z^{-6}U(z)$

$Y(z) = \left(\frac{1}{7} + \frac{1}{7}z^{-1} + \frac{1}{7}z^{-2} + \dots + \frac{1}{7}z^{-6}\right)U(z)$

$z^{6}Y(z) = \left(\frac{1}{7}z^{6} + \frac{1}{7}z^{5} + \frac{1}{7}z^{4} \dots + \frac{1}{7}\right)U(z)$

$H(z) = \dfrac{Y(z)}{U(z)} = \dfrac{\frac{1}{7}z^{6} + \frac{1}{7}z^{5} + \dots\dots + \frac{1}{7}}{z^{6}}$ - this is your discrete time transfer function.

<u>Exerpl.</u>

non-causal.

$y[k] = \frac{1}{7}u[k+3] + \frac{1}{7}u[k+2] + \frac{1}{7}u[k+1] + \frac{1}{7}u[k] + \frac{1}{7}u[k-1] + \frac{1}{7}u[k-2] + \frac{1}{7}u[k-3]$

$\mathcal{Z}$

$Y(z) = \Big(\underbrace{\frac{1}{7}z^{3} + \frac{1}{7}z^{2} + \frac{1}{7}z}_{\text{but output depends on values yet to come.}} + \frac{1}{z} + \frac{1}{z}z^{-1} + \frac{1}{7}z^{-2} + \frac{1}{7}z^{-3}\Big)U(z)$

$H(z) = \dfrac{Y(z)}{U(z)} = \dfrac{\frac{1}{7}z^{3} + \frac{1}{7}z^{2} + \frac{1}{7}z + \frac{1}{7} + \frac{1}{7}z^{-1} + \cancel{\frac{1}{7}z^{-2}}\ \frac{1}{7}z^{-3})}{1}$

$= \dfrac{\left(\frac{1}{7}z^{6} + \frac{1}{7}z^{5} + \frac{1}{7}z^{4} \dots\dots\dots + \frac{1}{z}\right)}{z^{3}}$

order of num. > den.

∴ so improper

∴ causality relate directly to properness of the system.

Back to causal.

$$H(z) = \frac{Y(z)}{U(z)} = \frac{\frac{1}{7}z^6 + \frac{1}{7}z^5 + \dots + \frac{1}{7}}{z^6}$$

proper
moving-average filter
(all poles at 0).

table 4.3. Properties of Z-transform.

• linearity

$$\alpha_1 f_1[k] + \alpha_2 f_2[k] \longleftrightarrow \alpha_1 F_1(z) + \alpha_2 F_2(z)$$

• Time-delay

$$g[k] = f[k-1]$$
$$G[z] = z^{-1}F(z)$$

• Time-advance.

$$g[k] = f[k+1]$$
$$G(z) = zF(z)$$

• Convolution in time domain. $\leftrightarrow$ Multiplication in Z domain.

$$f_1[k] \circledast f_2[k] \longleftrightarrow F_1(z)F_2(z)$$

convolution is this.

$$= \sum_{i=0}^{k} f_1[k-i]\, f_2[i]$$
$$= \sum_{i=0}^{k} f_1[k]\, f_2[k-i]$$

Linear Difference Equation-

$$y[k] + a_1 y[k-1] + \dots + a_n y[k-n] = b_0 u[k] + b_1 u[k-1] + \dots + b_n u[k-n]$$

Z

$$(1 + a_1 z^{-1} + \dots + a_n z^{-n}) Y(z) = (b_0 + b_1 z^{-1} + \dots + b_n z^{-n}) U(z)$$

$$(z^n + a_1 z^{n-1} + \dots + a_n) Y(z) = (b_0 z^n + b_1 z^{n-1} + \dots + b_n) U(z)$$

transfer function.

$$H(z) = \frac{Y(z)}{U(z)} = \frac{(b_0 z^n + b_1 z^{n-1} + \dots b_n)}{(z^n + a_1 z^{n-1} + \dots a_n)}$$

z-dom. ~~auto~~

$$Y(z) = H(z)\, U(z)$$

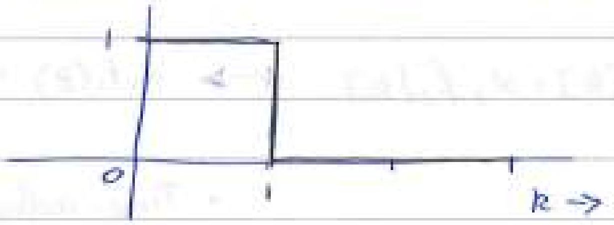
Z-domain output

$$Y(z) = H(z)\,U(z)$$

like w/ CT systems we can obtain $Y(z) = H(z)$ by choosing $U(z) = 1$.

what $u[k]$ has z-transform $U(z) = 1$?

DT Impulse (Kronecker delta) $\delta[k] = \begin{cases} 1 & k=0 \\ 0 & \text{else} \end{cases}$



<u>Break</u>.

let a system be governed by

$$y[k] - 2y[k-1] - 3y[k-2] = u[k-2]$$

$$\mathcal{Z}: \quad (1 - 2z^{-1} - 3z^{-2})\,Y(z) = z^{-2}\,U(z)$$

$$H(z) = \frac{Y(z)}{U(z)} = \frac{z^{-2}}{1-2z^{-1}-3z^{-2}} = \frac{1}{z^2-2z-3}$$

What is the impulse response?

output corresponding to input $u[k] = \delta[k]$

hence $Y(z) = H(z)\underbrace{U(z)}_{=1} = H(z) = \dfrac{1}{z^2-2z-3}$

→ Do PFE on $\dfrac{Y(z)}{z}$ for D.T.

but no way to convert so....

to then expand, ~~convert~~ then return back to normal, then transform to $h[k]$

Ex.: $\frac{H(z)}{z} = \frac{1}{z^3 - 2z^2 - 3z} = \frac{A_0}{z} + \frac{A_1}{z+1} + \frac{A_2}{z-3}$

extra pole at 0.!

$$A_0 = H(0) = -\frac{1}{3}$$

$$A_1 = \frac{H(z)}{z}(z+1)\Big|_{z=-1} = \frac{\cancel{(z+1)}\,1}{z\cancel{(z+1)}(z-3)}\Big|_{z=-1} = \frac{1}{4}$$

$$A_2 = \frac{H(z)}{z}(z-3)\Big|_{z=3} = \frac{\cancel{(z-3)}\,1}{z(z+1)\cancel{(z-3)}}\Big|_{z=3} = \frac{1}{12}$$

$$\frac{H(z)}{z} = \frac{-\frac{1}{3}}{z} + \frac{1/4}{(z+1)} + \frac{1/12}{(z-3)}$$

$$H(z) = -\frac{1}{3} + \frac{1}{4}\frac{z}{(z+1)} + \frac{1}{12}\frac{z}{(z-3)}$$

$\mathcal{Z}^{-1}$

{ from table }

$$h[k] = -\frac{1}{3}\delta[k] + \frac{1}{4}(-1)^k + \frac{1}{12}(3)^k$$

→ Generic (omitting repeated poles)

$$H(z) = \frac{b(z)}{(z-p_1)(z-p_2)\cdots(z-p_n)}$$

$$\frac{H(z)}{z} = \frac{A_0}{z} + \frac{A_1}{z-p_1} + \frac{A_2}{z-p_2} + \cdots + \frac{A_n}{z-p_n}$$

$$H(z) = A_0 + A_1\frac{z}{z-p_1} + A_2\frac{z}{(z-p_2)} + \cdots + A_n\frac{z}{(z-p_n)}$$

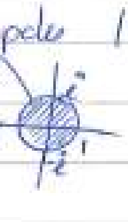
$\mathcal{Z}^{-1}$

$$h[k] = A_0\delta[k] + A_1(p_1)^k + A_2(p_2)^k + \cdots + A_n(p_n)^k$$

when stable BIBO $\|h[k]\| < \infty$

as long as pole $|P_k| < 1 \ \forall k$ Stable

one in this region they are stable.



complex values of P_k

DT.

$$P_k = e^{i\pi}$$

$$(P_k)^k = e^{i\pi k}$$

most cases you'll be within the unit circle not on it.

Mathematical Modelling Lecture 6.

So far, ◦ high order differential equations.

$u, u', u'', \dots, u^{(n)}$ $\quad a_1, \dots a_n, b_0, b_1, \dots, b_n$

$y, y', y'', \dots, y^{(n)}$

◦ transfer function.

$$\frac{b_0 s^n + b_1 s^{n-1} + \dots + b_n}{s^n + a_1 s^{n-1} + \dots + a_n}$$

$$\updownarrow \mathcal{L}$$

◦ Impulse responce $\quad h(t)$

- state-space system $\quad \begin{cases} x'(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}$

$x(t) \in \mathbb{R}^n$

$$f(t) \to \int_0^\infty f(t) e^{-st} dt = F(s) \qquad af(t) \xrightarrow{\mathcal{L}} aF(s)$$

$$\begin{pmatrix} f_1(t) \\ \vdots \\ f_n(t) \end{pmatrix} \to \begin{pmatrix} \int_0^\infty f_1(t) e^{-st} dt \\ \vdots \\ \int_0^\infty f_n(t) e^{-st} dt \end{pmatrix} = F(s)$$

also a vector function.

$$Af(t) \xrightarrow{\mathcal{L}} AF(s)$$

Now apply laplace to the ss system.

$x'(t) = Ax(t) + Bu(t)$

$y(t) = Cx(t) + Du(t)$

$\downarrow \mathcal{L}$

$\begin{cases} sX(s) = AX(s) + BU(s) \\ Y(s) = CX(s) + DU(s) \end{cases}$

$X(s) = (sI - A)^{-1} BU(s)$

$Y(s) = [C(sI-A)^{-1}B + D]\, U(s)$

$Y(s) = \quad H(s) \quad U(s)$

$\Rightarrow \quad sX(s) - AX(s) = BU(s)$

$\Rightarrow \quad (sI - A) X(s) = BU(s)$

$X(s) = (sI - A)^{-1} BU(s)$

$Y(s) = CX(s) + DU(s)$

$= C(sI-A)^{-1}BU(s) + DU(s)$

$$\boxed{Y(s) = [\overbrace{C(sI-A)^{-1}B + D}^{H(s)}] + U(s)}$$

$$H(s) = C(sI-A)^{-1}B + D$$ transfer func.

Ex. $A = \begin{pmatrix} 2 & -1 \\ 4 & 5 \end{pmatrix}$ $\quad sI - A = \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} - \begin{pmatrix} 2 & -1 \\ 4 & 5 \end{pmatrix}$

$$= \begin{pmatrix} s-2 & +1 \\ -4 & s-5 \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$\Downarrow$ inversion.

$$(sI-A)^{-1} = \frac{\begin{pmatrix} s-5 & -1 \\ 4 & s-2 \end{pmatrix}}{(s-2)(s-5)+4}$$

degree of numerator ≤ 2
" " denominator $= 2$

$C = (\cdots\cdots)$ row vec $\quad B = \begin{pmatrix} \cdot \\ \vdots \\ \cdot \end{pmatrix}$ col vec.

denominator of transfer function.

$= \det(sI-A)$

characteristic polynomial of A.

$$(sI-A)^{-1} = \frac{\operatorname{adj}(sI-A)}{\det(sI-A)}$$

zeros of the denominator
=
eigen values of A
=
poles of $H(s)$

eigenvalues of A determine the stability of a state-space system.

$$\boxed{H(s) = \frac{b_0 s^n + \cdots + b_n}{s^n + a_1 s^{n-1} + \cdots + a_n}}$$

2n+1 parameters

statespace (A,B,C,D)

$$H(s) = C(sI-A)^{-1}B + D.$$

$(A)_{n\times n}$ $\quad (B)_{n\times 1}$

$(C)_{n\times 1}$ $\quad (D)_{1\times 1}$

extra parameters left of freedom.

$n^2 + 2n + 1$

parameters

system
$$x'(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

new state vector: $\bar{x}(t) = Tx(t)$ T : n×n matrix invertible.

$$\bar{x}'(t) = Tx'(t)$$

$$\bar{x}'(t) = Tx'(t) = TAx(t) + TBu(t)$$ $x(t) = T^{-1}\bar{x}(t)$

$$\boxed{\begin{aligned} \bar{x}'(t) &= (TAT^{-1})\bar{x}(t) + (TB)u(t). \\ y(t) &= (CT^{-1})\bar{x}(t) + Du(t). \end{aligned}}$$

new linear state space with changed coordinates of $\bar{x}(t)$ instead of $x(t)$

so the transfer func. must be ~~unchanging~~ unchanged.

$$C(sI-A)^{-1}B+D = C\underbrace{T^{-1}(sI-TAT^{-1})^{-1}T}B+D$$ $T = (T^{-1})^{-1}$

$$C\left(T^{-1}(sI-TAT^{-1})T\right)^{-1}B+D.$$

$(PQR)^{-1} = R^{-1}Q^{-1}P^{-1}$

T^{-1} $(sI-TAT^{-1})^{-1}$ $T = (T^{-1})^{-1}$

$$\underbrace{sT^{-1}IT}-A$$
$$\boxed{sI - A}$$

so here we show even if we do a linear transformation on our state the transfer function is still the same in general.

$(PQR)^{-1}$
$= (T^{-1}*(sI-TAT^{-1})*T)^{-1}$
$= (T^{-1}sIT - T^{-1}TAT^{-1}T)^{-1}$
$= (sI-A)^{-1}$

from a transfer function to a state-space system.
(general given transfer func.)

$$H(s) = \frac{b_0 s^n + \dots + b_n}{s^n + a_1 s^{n-1} + \dots + a_n} \longrightarrow (A,B,C,D)$$
s.t.
$$C(sI-A)^{-1}B+D = H(s)$$

column of -ive a_i's   add I

$$A = \left(\begin{array}{c|cccc} -a_1 & 1 & 0 & \dots & 0 \\ -a_2 & 0 & 1 & \dots & 0 \\ \vdots & & & \ddots & \vdots \\ -a_{n-1} & 0 & \dots & & 1 \\ \hline -a_n & 0 & \dots & & 0 \end{array}\right) \qquad B = \begin{pmatrix} b_1 - b_0 a_1 \\ b_2 - b_0 a_2 \\ \vdots \\ b_n - b_0 a_n \end{pmatrix}$$

add row of zeros   subtract $b_0 a_i$ col. of b_i's

$$C = (1 \mid 0 \dots 0) \qquad D = b_0$$

just always b_0.

→ observable canonical form.

Ex. to show it works.

$\underbrace{c(sI-A)^{-1}}B+D.$

-1's on off diagonal.

$$(1\ 0\ \ldots\ 0)\begin{pmatrix} s+a_1, -1, & & 0 \\ a_2, s, & \ddots & \\ \vdots & & \ddots\ -1 \\ a_n & 0 & s \end{pmatrix}^{-1} = (\omega_1\ \omega_2\ \ldots\ \omega_n)$$

-1 on off diagonal.

$$(1\ 0\ \ldots\ 0,0) = (\omega_1\ \omega_2\ \ldots\ \omega_n)\begin{pmatrix} s+a_1, -1, & & 0 \\ a_2, s, -1 & \ddots & \\ \vdots & & \ddots \\ a_n & 0 & s \end{pmatrix}$$

$*_3\ *_1\ *_1$

$(\underbrace{s^{n-1}, s^{n-2}}_{*_3}\ \ldots\ s^2, s, 1)$ $\quad 1 = s^0$ $\quad *_2 \quad *_1$

$$(s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots + a_{n-1}s + a_n)\cdot(1\ 0\ \ldots\ 0).$$

$$(\omega_1\ \ldots\ \omega_n) = \frac{(s^{n-1}, s^{n-2}, \ldots, s, 1)}{s^n + a_1 s^{n-1} + \ldots + a_{n-1}s + a_n}$$

now you multiply the above by B to finally produce the $c(sI-A)^{-1}B$. the $b_i - b_0a_i$ cancels out with the a values to produce the $c(sI-A)^{-1}B$ a we've defined before.

$$H(s) = \frac{b_0 s^n + b_1 s^{n-1} + \ldots\ b_n}{s^n + a_1 s^{n-1} + \ldots + a_n}$$

→ controlable canonical form.

$$A = \begin{pmatrix} -a_1, & \ldots\ldots & , -a_n \\ 1 & \ddots\quad 0 & 0 \\ & \ddots & \vdots \\ 0 & \quad 1 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$C = (b_1 - b_0 a_1\ \ldots\ b_n - b_0 a_n) \qquad D = b_0$$

~~$H(s) = c(sI-A)^{-1}B+D$~~

$$\underset{1\times1}{H(s)} = C(sI-A)^{-1}B + \underset{1\times1}{D}$$

transpose $\downarrow$

$$H(s) = \underset{\text{see before}}{B^T}\underbrace{\left((sI-A)^{-1}\right)^T}\underset{\text{see before}}{C^T} + D$$

$$(M^{-1})^T = (M^T)^{-1}$$
(for m×m matrices)

$$= B^T\left((sI-A)^T\right)^{-1}C^T + D$$

$$H(s) = B^T(sI-A^T)^{-1}C^T + D$$

$$x[k+1] = \cancel{A[x+1]}\ Ax[k] + Bu[k]$$
$$y[k] = Cx[k] + Du[k]$$

$$Y(z) = H(z)\cdot U(z) \Rightarrow H(z) = \frac{b_0 z^n + \dots + b_n}{z^n + a_1 z^{n-1} + \dots + a_n}$$

$$= C(z-IA)^{-1}B + D$$

Example.

Ex17. pg 70 Nam. 4

$$x'(t) = \overset{A}{\begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}} x(t) + \overset{B}{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} u(t)$$

$$y(t) = \underset{C}{(1\ 0)}\, x(t) + \underset{D}{0}\cdot u(t)$$

$$H(s) = C(sI-A)^{-1}B + D$$

$$= (1\ 0)\begin{pmatrix} s-1 & 1 \\ -2 & s-1 \end{pmatrix}^{-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 0$$

$$= (1\ 0)\frac{\begin{pmatrix} s-1 & -1 \\ 2 & s-1 \end{pmatrix}}{(s-1)(s-1)-(-2)}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{s-1}{(s-1)^2+2} = \frac{s-1}{s^2-2s+3}$$

$$(sI-A)^{-1} = ? = \frac{\cdots\cdots}{\det(sI_n - A)}$$

$$\left(\begin{array}{ccc|ccc} s-6 & 7 & 3 & 1 & 0 & 0 \\ -1 & 5 & 11 & 0 & 1 & 0 \\ 0 & 2 & s+3 & 0 & 0 & 1 \end{array}\right)$$

S (but this is a pain in the ass).

$$(sI_n - A)^{-1} = ? = \frac{\cdots\cdots}{\det(sI_n - A)}$$

$$\det(sI_n - A) = a(s) = s^n + a_1 s^{n-1} + \ldots\ldots + a_{n-1}s + a_n.$$

$$S_1 = I_n \longrightarrow AS_1 = A = S_1 A \qquad a_1 = -\mathrm{trace}(AS_1)/1$$

trace is the sum of entries on main diagonal.

$$S_2 = AS_1 + a_1 I_n \rightarrow AS_2 \qquad a_2 = -\mathrm{trace}(AS_2)/2$$

$$S_3 = AS_2 + a_2 I_n \rightarrow AS_3 \qquad a_3 = -\mathrm{trace}(AS_3)/3$$

$\vdots$

$$S_{n-1} = AS_{n-2} + a_{n-2} I_n \rightarrow AS_{n-1} \qquad a_{n-1} = -\mathrm{trace}(AS_{n-1})/(n-1)$$

$$S_n = AS_{n-1} + a_{n-1} I_n \rightarrow AS_n \qquad a_n = \cancel{S_n} - \frac{\mathrm{trace}(AS_n)}{n}$$

$$(sI-A)^{-1} = \frac{S_1 s^{n-1} + S_2 s^{n-2} + \ldots + S_{n-1}s + S_n}{s^n + a_1 s^{n-1} + \ldots\ldots + a_n}$$

Note $S_{n+1} = AS_n + a_n I_n = 0$

can be used as a check, if
you don't get zero at S_{n+1}
you know you fucked up somewhere.

$S_1 = I_n$

$S_2 = AS_1 + a_1 I_n$

$S_3 = AS_2 + a_2 I_n$

$\vdots$

$S_n = AS_{n-1} + a_{n-1} I_n$

$S_{n+1} = AS_n + a_n I_n$

$S_1 = I$

$S_2 = A + a_1 I$

$S_3 = A^2 + a_1 A + a_2 I$

3×3

for n×n matrix

I_n

$A + a_1 I_n$

$A^2 + a_1 A + a_2 I_n$

$A^3 + a_1 A^2 + a_2 A + a_3 I_n$

$\vdots$

$$S_{n+1} = A^n + a_1 A^{n-1} + a_2 A^{n-2} + \dots + a_{n-1} A + a_n I_n = 0.$$

$s^n + a_1 s^{n-1} + \dots + a_{n-1} s + a_n$ $\quad s \to A$

back to example.

$$\begin{pmatrix} s-6 & 7 & 3 \\ -1 & 5 & 1 \\ 0 & 2 & s+3 \end{pmatrix}$$

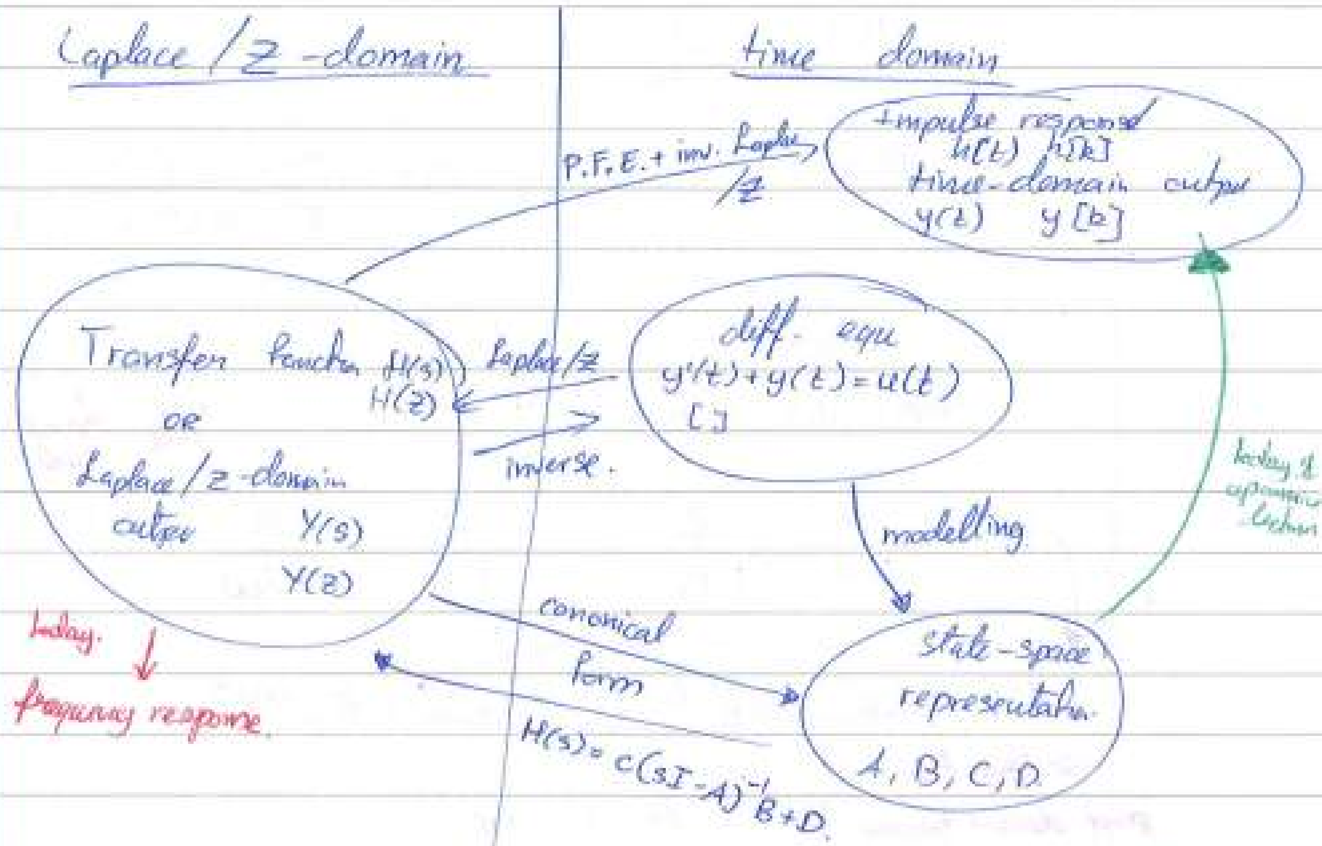
$S_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$a_1 = -\mathrm{trace}(AS_1)/1$

$= -\mathrm{trace}\begin{pmatrix} s-6 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & s+3 \end{pmatrix}$

$a_1 = -5(s-6)(s+3).$

Mathematical Modelling
Lecture 7 Sinusoidal Fidelity.



Sinusoidal Fidelity

$$u(t) \rightarrow \boxed{H(s)} \rightarrow y(t) \qquad\qquad Y(s) = H(s)U(s)$$

input

$$u(t) = \begin{cases} \sin(\omega t) \\ \cos(\omega t) \\ e^{i\omega t} = \cos(\omega t) + i\sin(\omega t). \end{cases}$$

ω angular frequency (Rad/s)

$\sin\left(\frac{2\pi}{T} t\right) \qquad \omega = \frac{2\pi}{T}$

$T = \frac{2\pi}{\omega}$ period.

What is the output for a stable system in Steady-state?

$$Y(s) = H(s)U(s)$$

$$u(t) = e^{i\omega t} \qquad e^{i\omega t} \xrightarrow{\mathcal{L}} \frac{1}{s-a}$$

$$U(s) = \frac{1}{s-i\omega}$$

$$Y(s) = H(s)U(s) = \frac{b(s)}{a(s)}U(s) = \frac{b_0 s^n + b_1 s^{n-1} \dots + b_n}{s^n + a_1 s^{n-1} + \dots + a_n} \frac{1}{s-i\omega}$$

$$= \frac{b(s)}{(s-p_1)(s-p_2)\dots(s-p_n)} \cdot \frac{1}{s-i\omega} \qquad \mathrm{Re}\{p_k\} < 0 \quad \forall_k$$

real part — of all poles — -ive

$$Y(s) = \frac{A_1}{s-p_1} + \frac{A_2}{s-p_2} + \dots + \frac{A_n}{s-p_n} + \frac{B}{s-i\omega}$$

$$\mathcal{L}^{-1} \downarrow$$

$$y(t) = \underbrace{A_1 e^{p_1 t} + A_2 e^{p_2 t} + \dots + A_n e^{p_n t}}_{\Rightarrow 0 \text{ for } t \to \infty} + B e^{i\omega t}$$

so then this part doesn't become relevant for ~~sinusoidal~~ stability.

How to compute B?

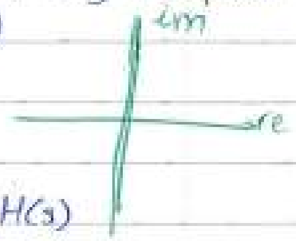
$$Y(s)(s-i\omega)\Big|_{s=i\omega} = \frac{A_1(s-i\omega)}{s-p_1}\Big|_{s=i\omega} + \frac{A_2(s-i\omega)}{s-p_2}\Big|_{s=i\omega} + \dots + \frac{A_n(s-i\omega)}{s-p_n}\Big|_{s=i\omega} + B$$

$$B = Y(s)(s-i\omega)\Big|_{s=i\omega}$$

$$B = H(s)U(s)(s-i\omega)\Big|_{s=i\omega}$$

$$= H(s)\frac{1}{\cancel{s-i\omega}}\cancel{(s-i\omega)}\Big|_{s=i\omega} = H(i\omega)$$

frequency response

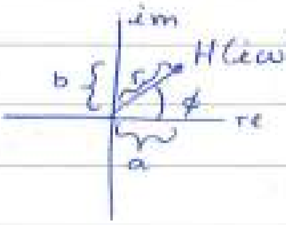


$$h(t) \xrightarrow{\mathcal{L}} \int_0^\infty e^{-st} h(t)\,dt = H(s)$$

$$h(t) \xrightarrow{\mathcal{F}} \int_{-\infty}^{\infty} e^{-i\omega t} h(t)\,dt = H(\omega) = H(i\omega)$$

Input - $u(t) = e^{i\omega t}$

steady-state output $y_s(t) = H(i\omega)e^{i\omega t}$



$H(i\omega) = a + bi = re^{i\phi}$

$r = \sqrt{a^2+b^2}$

$y_s(t) = re^{i\phi}e^{i\omega t}$

$= re^{i(\omega t + \phi)}$

$= |H(i\omega)|e^{i(\omega t + \angle H(i\omega))}$

$\phi = \text{atan2}(b,a)$

$$= \begin{cases} \arctan(b/a) & \text{if } a > 0 \\ \arctan(b/a) + \pi & \text{if } a < 0 \text{ and } b \geq 0 \\ \arctan(b/a) - \pi & \text{if } a < 0 \text{ and } b < 0 \\ \pi/2 & \text{if } a = 0 \text{ and } b > 0 \\ -\pi/2 & \text{if } a = 0 \text{ and } b < 0 \\ \text{undefined} & \text{if } a = 0 \text{ and } b = 0. \end{cases}$$

exercise 22

$H(s) = \dfrac{s^2 - 3s + 2}{s^2 + 4s + 3}$

$u(t) = 5\cos(3t)$ — $\cos(\omega t)$, $\omega = 3$

$y_s(t) = 5|H(3i)|(\cos(3t) + \angle H(3i))$

$y_s(t) = 5(0.85) \cdot \cos(3t + 2.0169)$

$y_s(t) = 4.25\cos(3t + 2.0169)$

$H(3i) = \dfrac{-9 - 9i + 2}{-9 + 12i + 3}$ — put $3i$ into $H(s)$

$= \dfrac{-7 - 9i}{-6 + 12i} \cdot \dfrac{-1 - 2i}{-1 - 2i}$ — multiply by complex conjugate

$= \dfrac{7 + 9i + 14i - 18}{6 - \cancel{12i} + \cancel{12i} + 24}$

$= \dfrac{-11 + 23i}{30}$ — a, b

$\angle H(3i) = \text{atan2}\left(\frac{23}{30}, \frac{-11}{30}\right)$

$= \text{atan2}(23, -11)$

$= \arctan(23, -11) + \pi \approx 2.0169$

$|H(3i)| = \sqrt{\left(\frac{-11}{30}\right)^2 + \left(\frac{23}{30}\right)^2}$

$= \sqrt{\frac{13}{18}} \approx 0.85$

Discrete-time sinusoidal Fidelity

$u[k] = e^{i\Omega k} = \cos(\Omega k) + i\sin(\Omega k)$

$\xrightarrow{z}$ $U(z) = \dfrac{z}{z - e^{i\Omega}}$

$H(z) = \dfrac{b(z)}{a(z)} = \dfrac{b(z)}{(z-p_1)(z-p_2)\cdots(z-p_n)}$

$$z^{-1}\quad Y(z) = H(z)U(z) = A_0 + \frac{zA_1}{(z-p_1)} + \frac{zA_2}{(z-p_2)} + \dots + \frac{zA_n}{(z-p_n)} + \frac{zB}{z-e^{i\Omega}}$$

becomes zero

$$y[k] = A_0\,\delta[k] + A_1(p_1)^k + A_2(p_2)^k + \dots + A_n(p_n)^k + B e^{i\Omega k}$$

if stable system, $\Rightarrow 0$ for $k \to \infty$

$|p_n|<1\ \forall p_n$

$$y_s[k] = B e^{i\Omega k}$$

$$Y(z) = \frac{zB}{z-e^{i\Omega}}$$

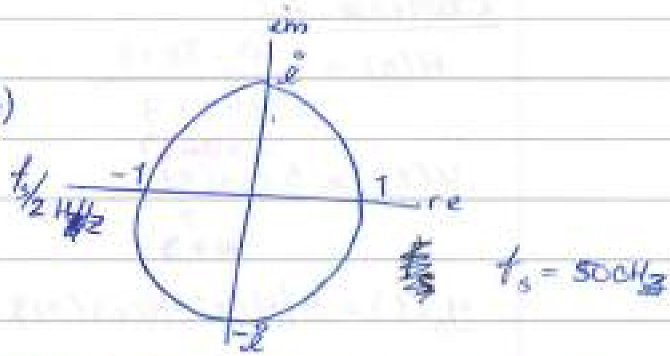
$$H(z)U(z) = H(z)\frac{\cancel{z}}{z-e^{i\Omega}} = \frac{\cancel{z}B}{z-e^{i\Omega}}$$

$$B = H(z)\cdot\frac{1}{\cancel{z-e^{i\Omega}}}\cdot\cancel{(z-e^{i\Omega})}\Big|_{z=e^{i\Omega}}$$

$$B = H(e^{i\Omega})$$

steady-state output discrete time

$$y_s[k] = |H(e^{i\Omega})|\, e^{i(k\Omega + \angle H(e^{i\Omega}))}$$



How to do filtering?

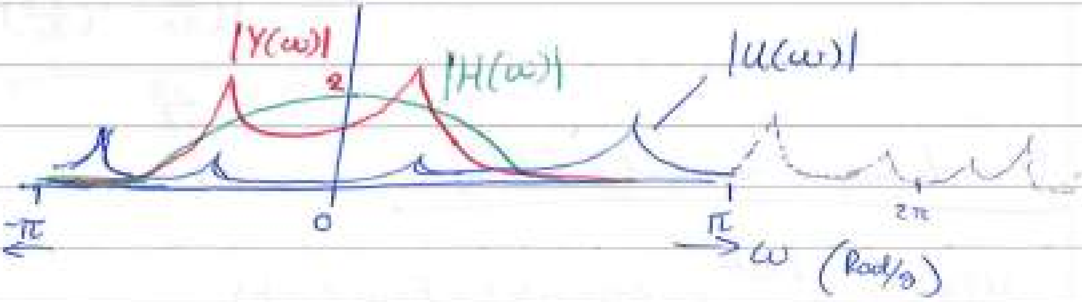
$u(t) \to \boxed{H(s)} \to y(t)$

$Y(s) = H(s)U(s)$

$Y(\omega) = H(\omega)U(\omega)$

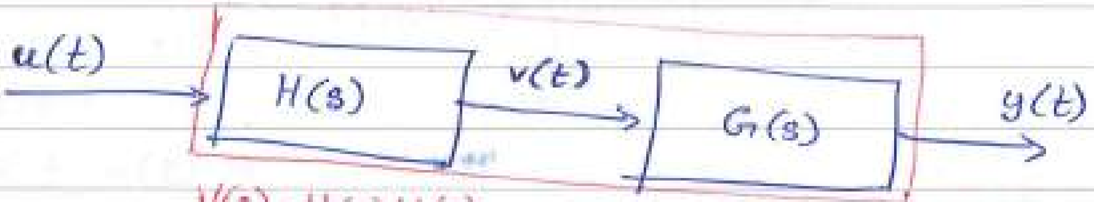
$\to U(s) = \int_0^{\infty} u(t)e^{-st}\,dt$

$\to U(\omega) = U(i\omega) = \int_{-\infty}^{\infty} u(t)e^{-i\omega t}\,dt$



MM Lecture 8 - Interconnected Systems.

- series / cascade interconnection.

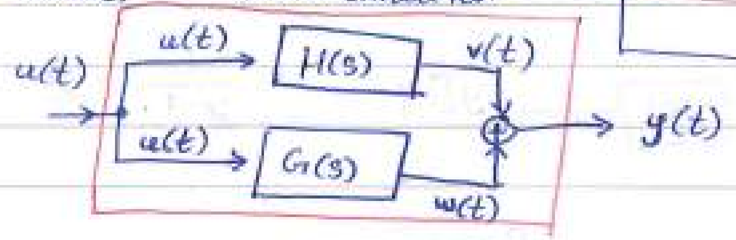


$V(s) = H(s)\,U(s)$

$Y(s) = G(s)\,V(s)$

$Y(s) = \boxed{G(s)H(s)}\,U(s)$ t.f. of overall system.

- parallel interconnection



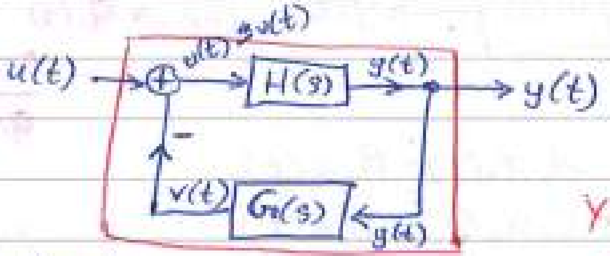
$$\boxed{\begin{aligned} V(s) &= H(s)\,U(s) \\ W(s) &= G(s)\,U(s) \\ Y(s) &= V(s) + W(s) \end{aligned}}$$

$Y(s) = H(s)\,U(s) + G(s)\,U(s)$

$\boxed{Y(s) = (H(s) + G(s))\,U(s)}$

$\underline{H(s) + G(s)}$

- feedback interconnection



~~networks~~.

$Y(s) = H(s)(U(s) - V(s))$

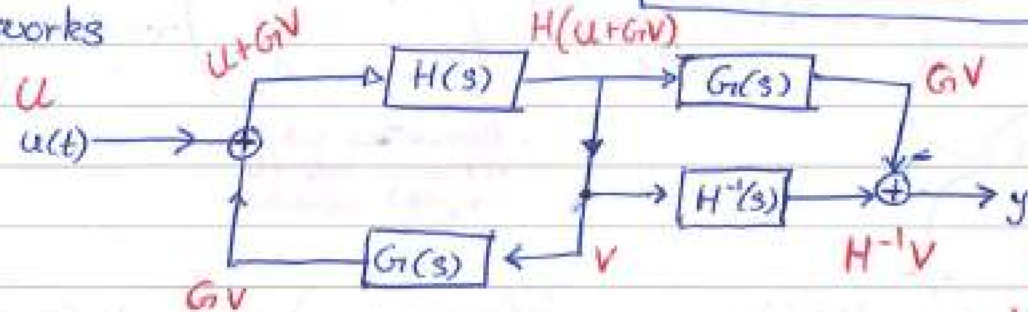
$V(s) = G(s)\,Y(s)$

$Y(s) = H(s)(U(s) - G(s)\,Y(s))$

$(1 + H(s)G(s))Y(s) = H(s)\,U(s)$

$$Y(s) = \boxed{\frac{H(s)}{(1 + H(s)G(s))}} \cdot U(s)$$

- networks



$Y = H^{-1}V - GV$

$$Y = \boxed{\frac{(H^{-1} - G)H}{1 - HG}}\,U$$

$V = H(U + GV)$ $\boxed{GH = HG}$

$V - HGV = HU$

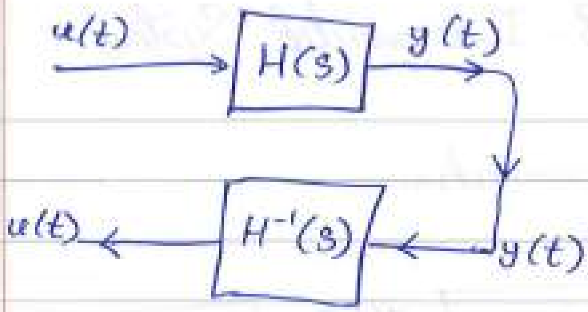
$(1 - HG)V = HU$

$$\boxed{V = \frac{H}{1 - HG} \cdot U}$$

$$\frac{1 - GH}{1 - HG} = 1$$

so by chance

$Y = U$



inverse transfer function.

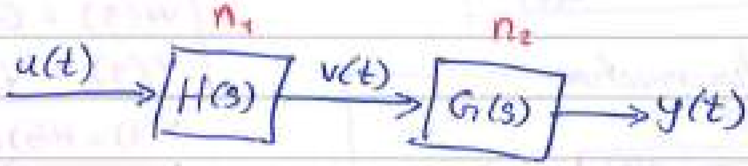
$$H^{-1}(s) = \frac{1}{H(s)}$$

$$H(s) = \frac{P(s)}{q(s)}$$

$$H^{-1}(s) = \frac{q(s)}{P(s)}$$

$$y[k] = u[k-1]$$

$$\frac{1}{Z^{-1}} = Z \qquad \frac{Z}{1}$$



$$H(s) = \frac{P_1(s)}{Q_1(s)} \qquad n_1 \to Q_1(s)$$

$$G(s) = \frac{P_2(s)}{Q_2(s)} \qquad n_2$$

$$G(s)H(s) = \frac{P_1(s)P_2(s)}{Q_1(s)Q_2(s)}$$

$H(s)$: (A_1, B_1, C_1, D_1)

$$x_1'(t) = A_1 x_1(t) + B_1 u(t)$$
$$v(t) = C_1 x_1(t) + D_1 u(t)$$

$G(s)$: (A_2, B_2, C_2, D_2)

$$x_2'(t) = A_2 x_2(t) + B_2 v(t)$$
$$y(t) = C_2 x_2(t) + D_2 v(t)$$

$$x(t) = \begin{pmatrix} x_1(t) \\ \hline x_2(t) \end{pmatrix} \begin{matrix} \updownarrow n_1 \\ \updownarrow n_2 \end{matrix}$$

$G(s)H(s)$

$$x'(t) = \begin{pmatrix} x_1'(t) \\ \hline x_2'(t) \end{pmatrix} = \overbrace{\begin{pmatrix} A_1 & 0 \\ B_2 C_1 & A_2 \end{pmatrix}}^{A} x(t) + \overbrace{\begin{pmatrix} B_1 \\ B_2 D_1 \end{pmatrix}}^{B} u(t) \qquad \underline{n_1 + n_2}$$

because we sub $v(t) = \ldots$ into the $x_2'(t)$ equation.

$$y(t) = \overbrace{\begin{pmatrix} D_2 C_1 & C_2 \end{pmatrix}}^{C} x(t) + \overbrace{(D_2 D_1)}^{D} u(t)$$

order $\leq n_1 + n_2$

(need to cancel like terms in transfer func.)

here to scale the leading coefficient.

$$H(z) = C(zI-A)^{-1}B + \overset{b_0}{\underset{\|}{D}} = \frac{D \, \det(zI-(A-BD^{-1}C))}{\det(zI-A)}$$

$$H^{-1}(z) = ? -D^{-1}C(zI-(A-BD^{-1}C))^{-1}BD^{-1} + D^{-1}$$

$$x[k+1] = Ax[k] + Bu[k]$$
$$y[k] = Cx[k] + Du[k]$$

$$x[k+1] = (A-BD^{-1}C)x[k] + (BD^{-1})y[k]$$
$$u[k] = (-D^{-1}C)x[k] + (D^{-1})y[k]$$

$$\frac{1}{D}y[k] - \frac{C}{D}x[k] = \cancel{D}u[k]$$

$$x[k+1] = Ax[k] + B(-D^{-1}Cx[k]) + BD^{-1}y[k]$$

Solving State Equations

$$x'(t) = Ax(t) + Bu(t)$$
$$x(t) = ? \qquad x(0) = x_0$$

$$x[k+1] = Ax[k] + Bu[k]$$
$$x[k] = ? \qquad x[0] = x_0$$

$$x[0] = x_0$$
$$x[1] = Ax_0 + Bu[0]$$
$$x[2] = A^2x_0 + ABu[0] + Bu[1]$$
$$x[3] = A^3x_0 + A^2Bu[0] + ABu[1] + Bu[2]$$

$$\boxed{x[k] = A^k x_0 + \sum_{\ell=0}^{k-1} A^{k-1-\ell}Bu[\ell]}$$

$$y[k] = Cx[k] + Du[k]$$
$$y[k] = CA^kx_0 + \sum_{\ell=0}^{k-1} CA^{k-1-\ell}Bu[\ell] + Du[k]$$

$$h[k] = \begin{cases} D & k=0 \\ CA^{k-1}B & k>0 \end{cases}$$ impulse response

$$\boxed{x'(t) = Ax(t)} \qquad x(0) = x_0$$

$$\dot{x} = ax \qquad x(0) = x_0$$

$$x(t) = e^{at} x_0$$

$$\boxed{x(t) = e^{At} x_0}$$

Matrix exponential

$$e^{x} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$e^{M} = I + M + \frac{M^2}{2!} + \frac{M^3}{3!} + \dots$$

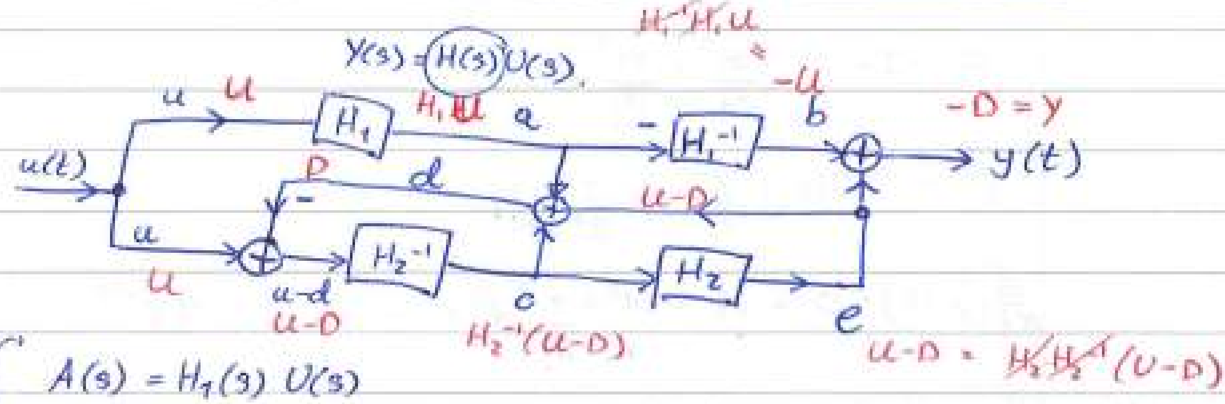
$$\frac{d}{dt}\left(e^{At} x_0\right) = Ae^{At} x_0 = Ax(t)$$

$$e^{At} = I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \dots$$

$$\frac{d}{dt}\left(e^{At}\right) = 0 + A + tA^2 + \frac{t^2}{2!}A^3 + \dots = A\left(e^{At}\right) = e^{At} \cdot A$$

Discrete-time equivalents of continuous systems.

Ex. 21
Q4.



simple way.
$$\begin{cases} A(s) = H_1(s)\,U(s) \\ B(s) = H_1(s)^{-1}(-A(s)) \\ C(s) = H_2(s)^{-1}(U(s) - D(s)) \\ E(s) = H_2(s)(C(s)) \end{cases}$$

$$Y(s) = B(s) + E(s)$$
$$D(s) = A(s) + C(s) + E(s)$$

quick way.
$$D = H_1 U + U - D + H_2^{-1}(U - D)$$
$$D = H_1 U + U - D + H_2^{-1}U - H_2^{-1}D$$
$$(2 + H_2^{-1})D = (H_1 + 1 + H_2^{-1})U$$
$$Y = -D = -\underbrace{\frac{(H_1 + 1 + H_2^{-1})}{(2 + H_2^{-1})}}_{\text{this is gonna be our transfer func.}} U$$

---

discrete-time

$$x[k+1] = Ax[k] + Bu[k] \qquad x[0] = x_0.$$

$$x[0] = x_0$$
$$x[1] = Ax_0 + Bu[0]$$
$$x[2] = A^2 x_0 + ABu[0] + Bu[1]$$
$$x[3] = A^3 x_0 + A^2 Bu[0] + ABu[1] + Bu[2]$$
$$\vdots$$
$$x[k] = A^k x_0 + \sum_{l=0}^{k-1} A^{k-1-l} B u[l]$$
$$= A^k x_0 + \sum_{l=0}^{k-1} A^l B u[k-1-l]$$

same thing, different ways of writing index value.

scalar.
$$x'(t) = a\,x(t) \qquad x(0) = x_0$$
$$\boxed{x(t) = e^{at} x_0}$$

multi-variable
$$\underline{x}'(t) = A\underline{x}(t) \qquad \underline{x}(0) = \underline{x}_0$$
$$\boxed{\underline{x}(t) = e^{At}\underline{x}_0}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\dots$$

$$e^M = I + M + \frac{M^2}{2!} + \frac{M^3}{3!} + \dots\dots$$

$$e^{At} = I + At + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \dots\dots$$

$$\frac{d}{dt}(e^{At}) = 0 + A + tA^2 + \frac{t^2}{2!}A^3 + \dots\dots = Ae^{At}$$

$$\underline{x}'(t) = \frac{d}{dt}(e^{At}\underline{x}_0) = Ae^{At}\underline{x}_0 = A\cdot\underline{x}(t)$$

$$x'(t) = Ax(t) + Bu(t) \qquad x(0) = x_0$$

$$\boxed{x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau}$$

the current time we are integrating

$$x(t) = e^{At}x_0 + \int_0^t e^{At}Bu(t-\tau)\,d\tau$$

does it work?

$$x'(t) = Ae^{At}x_0 + e^{A(t-t)}Bu(t)\cdot 1 + \int_0^t Ae^{A(t-\tau)}Bu(\tau)\,d\tau$$

0 (we start at zero and differentiation of zero constant)

$$\frac{d}{dx}\left(\int_{a(x)}^{b(x)} f(x,t)\,dt\right) = f(x,b(x))\cdot\frac{d}{dx}b(x) - f(x,a(x))\cdot\frac{d}{dx}a(x) + \int_{a(x)}^{b(x)}\frac{\partial}{\partial x}f(x,t)\,dt$$

Leibniz's Rule.

$$x'(t) = A\underbrace{\left(e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau\right)}_{x(t)} + Bu(t)$$

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)} Bu(\tau)\,d\tau$$

$x(0) = x_0$
$x'(t) = Ax(t) + Bu(t)$
$y(t) = Cx(t) + Du(t)$

$$y(t) = Ce^{At}x_0 + \int_0^t Ce^{A(t-\tau)} Bu(\tau)\,d\tau + Du(t)$$

impulse response $u(t) = \delta(t)$ $x_0 = 0$

$$\boxed{h(t) = Ce^{At}B + D\delta(t)}$$

$\mathcal{L}$

$$H(s) = C(sI-A)^{-1}B + D$$

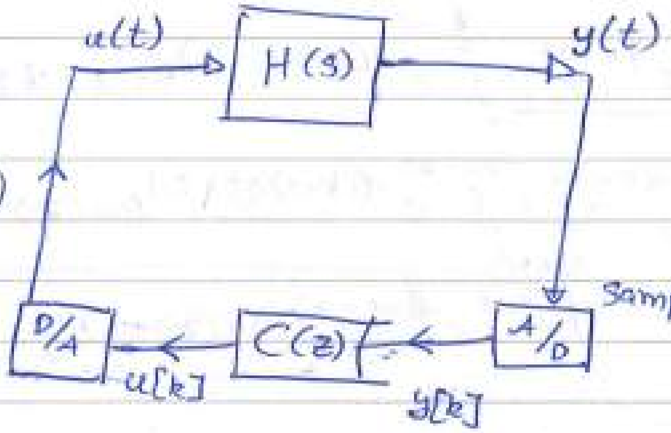
$\int_0^\infty f(t)\delta(t)\,dt = f(0)$

$\mathcal{L}\{\delta(t)\} = 1$

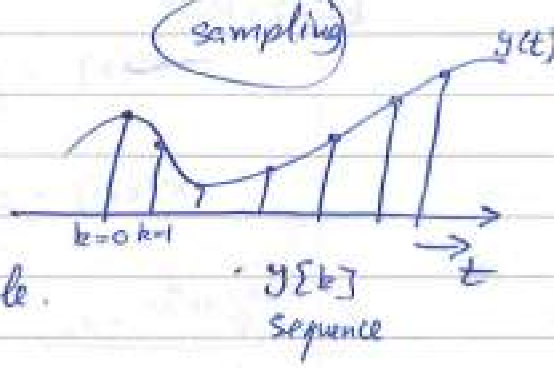
$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$

$\frac{1}{s-A}$

$((sI-A))^{-1}$



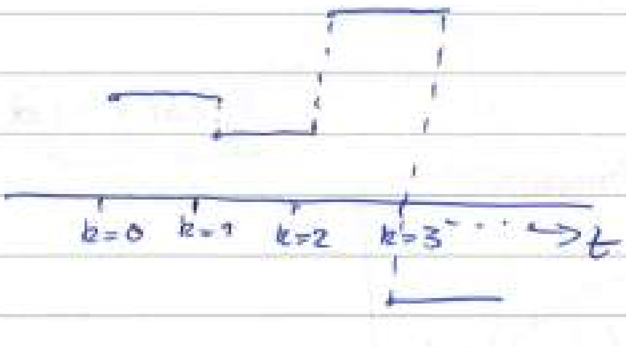
zero-order hold
ZOH.



piecewise constant

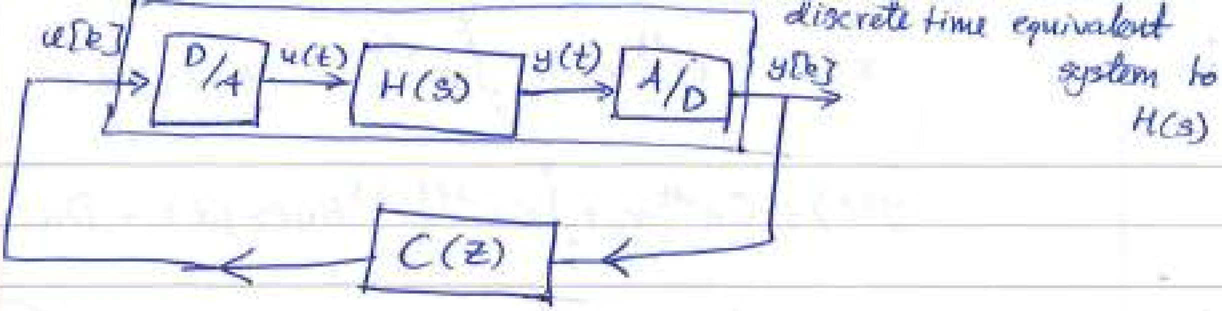
sampling interval Δt
T_s T

ZOH interpretation



$$\boxed{y[k] = y(k\cdot\Delta t)}$$

$u(t) = u[k]$ if $t \in [k\cdot\Delta t, (k+1)\Delta t)$



continuous-time: $\dot{x}(t) = Ax(t) + Bu(t)$ $\quad x(0) = x_0$

$$\boxed{y(t) = Cx(t) + Du(t)}$$

at k, $t = k\cdot\Delta t$ $\quad u[k] = u(k\cdot\Delta t)$

$$y[k] = y(k\cdot\Delta t)$$

$$x[k] = x(k\cdot\Delta t)$$

$$\boxed{y[k] = Cx[k] + Du[k]}$$

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau$$

$k \to t = k\Delta t$

$$x[k] = \underbrace{e^{Ak\cdot\Delta t}x_0}_{} + \int_0^{k\cdot\Delta t} e^{A(k\cdot\Delta t-\tau)}Bu(\tau)\,d\tau$$

$$x[k+1] = e^{A(k+1)\Delta t}x_0 + \underbrace{\int_0^{(k+1)\Delta t} e^{A((k+1)\Delta t-\tau)}Bu(\tau)\,d\tau}_{}$$

$$e^{A\Delta t}\underbrace{e^{Ak\Delta t}\cdot x_0}_{} \qquad \int_0^{k\Delta t} e^{A\Delta t}e^{A(k\Delta t-\tau)}Bu(\tau)\,d\tau$$

$$= e^{A\Delta t}x[k] + \int_{k\Delta t}^{k+1\Delta t} e^{A((k+1)\Delta t-\tau)}Bu(\tau)\,d\tau$$

$$\int_0^{\Delta t} e^{A(\Delta t-\tau)}\underbrace{Bu(k\Delta t+\tau)}_{u[k]}\,d\tau$$

$$\boxed{x[k+1] = e^{A\Delta t}x[k] + \left(\int_0^{\Delta t} e^{A(\Delta t-\tau)}\,d\tau\right)u[k]}$$

fixed matrix

fixed vector.

$$x[k+1] = \underset{\text{fixed matrix}}{\overset{\text{phi}}{\Phi}}\, x[k] + \underset{\text{fixed vector.}}{\overset{\text{gamma}}{\Gamma}}\, u[k]$$

$$\Phi = e^{A \cdot \Delta t} \qquad \Gamma = \int_0^{\Delta t} e^{A(\Delta t - \tau)} d\tau = \boxed{\int_0^{\Delta t} e^{A\tau'} d\tau'} \qquad \tau' = \Delta t - \tau$$

$$x'(t) = Ax(t) + Bu(t) \qquad (A, B, C, D)$$

$$\downarrow \Delta t \qquad\qquad \downarrow \Delta t$$

$$\Rightarrow \boxed{x[k+1] = \Phi x[k] + \Gamma u[k]} \qquad (\Phi, \Gamma, C, D)$$

this equation shows what happens in the controller.

example.

$$x'(t) = ax(t)$$

$$(y(t) = x(t))$$

$$x(t) = e^{at} x_0$$

$$x[k] = x(k\Delta t) = e^{ak\Delta t} x_0 = \left(e^{a\Delta t}\right)^k x_0$$

$$x[k+1] = \left(e^{a\Delta t}\right)^{k+1} x_0 = e^{A\Delta t} \cdot \left(e^{A\Delta t}\right)^k x_0$$

$$= e^{A\Delta t} \cdot x[k]$$

$$\boxed{x[k+1] = e^{a\Delta t} x[k]}$$

$$A \qquad\qquad \Phi = e^{A\Delta t}$$

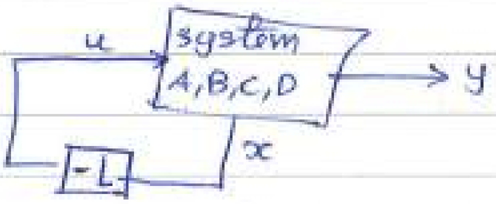
$$M \to e^M \qquad\qquad \lambda_1, \lambda_2, \ldots, \lambda_n \longrightarrow e^{\lambda_1 \Delta t}, e^{\lambda_2 \Delta t}, \ldots, e^{\lambda_n \Delta t}$$

$$\lambda = a + bi; \quad |e^{\lambda \Delta t}| = |e^{a\Delta t}| \cdot \underbrace{|e^{bi\Delta t}|}_{} = |e^{a\Delta t}| < 1$$

$$a < 0$$

Mathematical Modelling
Lecture 10

Pole-placement with state-feedback.



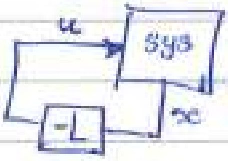
Can we automatically stabilize the system by choosing inputs using the state?

Yes! with fixed linear combination of states. (provided controllable).

Recall.

controllable if $W_c = [B \; AB \dots A^{n-1}B]$ is non-singular (is invertible/determinant $\neq 0$).

$$u = -Lx = [-l_1 x_1 \; -l_2 x_2 \dots -l_n x_n]$$



How to find L?

First for special case: sys in controllable canonical form.

CT

$$\begin{cases} x'(t) = Ax(t) + Bu(t) \\ \cancel{y(t) = Cx(t) + Du(t)} \\ u(t) = -Lx(t) \end{cases}$$

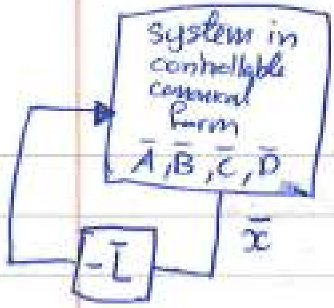
$$\Rightarrow x'(t) = Ax(t) - BLx(t) = (A - BL)x(t)$$

sys matrix of closed loop

DT.

$$\begin{cases} x[k+1] = Ax[k] + Bu[k] \\ u[k] = -Lx[k] \end{cases}$$

$$\Rightarrow x[k+1] = (A - BL)x[k]$$



$$\bar{A} = \left(\begin{array}{ccc|c} -a_1 & -a_2 & \cdots & -a_n \\ \hline & I & & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \end{array}\right)$$

$$\bar{B} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \qquad \bar{L} = [\bar{l}_1 \; \bar{l}_2 \; \ldots \; \bar{l}_n]$$

Then $$\bar{A} - \bar{B}\bar{L} = \left(\begin{array}{ccc|c} -a_1 & -a_2 & \cdots & -a_n \\ \hline & I & & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \end{array}\right) - \left(\begin{array}{c} \bar{l}_1 \; \bar{l}_2 \; \ldots \; \bar{l}_n \\ \hline 0 \end{array}\right)$$

$$= \left(\begin{array}{ccc|c} -a_1+\bar{l}_1 & -a_2+\bar{l}_2 & \cdots & -a_n+\bar{l}_n \\ \hline & I & & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \end{array}\right)$$

Characteristic Polynomial (CP) of $\bar{A}$.

$$\det(sI - \bar{A}) = s^n + a_1 s^{n-1} + \ldots + a_n.$$

Poles of the system.

- Roots of CP.
- eigenvalues of $\bar{A}$ $\quad P_1, P_2, \ldots, P_n$
- Poles

CP of $\bar{A} - \bar{B}\bar{L}$: $\det(sI - [\bar{A} - \bar{B}\bar{L}]) = s^n + (a_1 + \bar{l}_1)s^{n-1} + \ldots + (a_n + \bar{l}_n)$

desired pole location of closed loop system are $\tilde{P}_1, \tilde{P}_2, \ldots, \tilde{P}_n$

hence desired CP : $(s - \tilde{P}_1)(s - \tilde{P}_2) \ldots (s - \tilde{P}_n)$

$$= s^n + \tilde{a}_1 s^{n-1} + \tilde{a}_2 s^{n-2} + \ldots + \tilde{a}_n$$

hence we choose $\bar{l}_1 = \tilde{a}_1 - a_1$, $\bar{l}_2 = \tilde{a}_2 - a_2$ ... $\bar{l}_n = \tilde{a}_n - a_n$

$$\bar{L} = (\tilde{a}_1 - a_1, \; \tilde{a}_2 - a_2 \; \ldots \; \tilde{a}_n - a_n)$$

This is pole-placement.

What if the system is NOT in controllable canon. form?

1) pretend that it is in contr. canon. form and find $\bar{L}$.
2) find ss transformation T to bring it into controllable canonical form.
3) find the feedback law for the ss representation that we have.

$$\bar{A} = TAT^{-1} \qquad \bar{x}(t) = Tx(t)$$
$$\bar{B} = TB$$
$$\bar{C} = CT^{-1} \qquad T^{-1} = W_c \bar{W}_c^{-1}$$
$$\bar{D} = D \qquad = (B \; AB \ldots A^{n-1}B)\begin{pmatrix} 1 & a_1 & \cdots & a_n \\ & \ddots & \ddots & \vdots \\ & & \ddots & a_1 \\ 0 & & & 1 \end{pmatrix}$$

$$u(t) = -\bar{L}\bar{x}(t) = -\bar{L}Tx(t) = Lx(t)$$
$$\Rightarrow L = \bar{L}T$$

Ex.
$$\begin{cases} x'(t) = \begin{pmatrix} -2 & -2 & -\frac{3}{2} \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} u(t) \\ y(t) = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{3}{4} \end{pmatrix} x(t) \end{cases}$$

goal: all pole locations of closed-loop system at $s = -2$.

1) find $\bar{L}$

a) find CP

$$a(s) = \det(sI - A) = \det\begin{pmatrix} s+2 & 2 & 3/2 \\ -2 & s & 0 \\ 0 & -1 & s \end{pmatrix}$$
$$= s \det\begin{pmatrix} s+2 & 2 \\ -2 & s \end{pmatrix} + \det\begin{pmatrix} s+2 & 3/2 \\ -2 & 0 \end{pmatrix}$$
$$= s\left[(s^2+2s)+4\right] + 3 = s^3 + 2s^2 + 4s + 3$$
$$a_0 = 1, \; a_1 = 2, \; a_2 = 4, \; a_3 = 3$$

b) find desired CP.

we want pole at $s=-2$.
$$(s+2)^3 = s^3 + 6s^2 + 12s + 8$$
$$\hat{a}_0 = 1 \quad \hat{a}_1 = 6, \; \hat{a}_2 = 12, \; \hat{a}_3 = 8$$

c) find $\bar{L}$
$$\bar{l}_1 = \hat{a}_1 - a_1 = 6 - 2 = 4$$
$$\bar{l}_2 = \hat{a}_2 - a_2 = 12 - 4 = 8$$
$$\bar{l}_3 = \hat{a}_3 - a_3 = 8 - 3 = 5$$
$$\bar{L} = (4 \;\; 8 \;\; 5)$$

2) find T^{-1} and T.

$$W_c = (B \;\; AB \;\; A^2B)$$

$$= \begin{pmatrix} 2 & -4 & 0 \\ 0 & 4 & -8 \\ 0 & 0 & 4 \end{pmatrix}$$

$$\bar{W}_c^{-1} = \begin{pmatrix} 1 & a_1 & a_2 \\ 0 & 1 & a_1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

$$T^{-1} = W_c \bar{W}_c^{-1}$$

$$= \begin{pmatrix} 2 & -4 & 0 \\ 0 & 4 & -8 \\ 0 & 0 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

$$T^{-1} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix} \qquad T = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1/4 \end{pmatrix}$$

3) $L = \bar{L}T$

$$= (4 \;\; 8 \;\; 5)\begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1/4 \end{pmatrix}$$

$$= (2 \;\; 2 \;\; 5/4) = L.$$

$$x'(t) = \begin{pmatrix} -2 & -2 & -3/2 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} x(t) - \underbrace{\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}(2 \;\; 2 \;\; 5/4)}_{\text{produces } 3\times 3 \text{ matrix}} x(t)$$

you can then make one big 3x3 matrix
to show the closed loop state-space model for the system.

MM extra videos.

Controllability.

$$x[1] = Ax_0 + Bu[0]$$
$$x[2] = Ax[1] + Bu[1]$$
$$= A^2x_0 + ABu[0] + Bu[1]$$
$$\vdots$$
$$x[n] = A^n x_0 + A^{n-1}Bu[0] + \dots\dots + Bu[n-1]$$

these are all column vectors of length n

$$W_c = \left(b \,|\, Ab \,|\, \dots\dots \,|\, A^{n-1}b\right) = n \times n \text{ matrix}$$

controllability matrix.

(it's like everything in the expression for the state but in reverse order.)

input as vector $\tilde{u} = \begin{pmatrix} u[n-1] \\ \vdots \\ u[0] \end{pmatrix}$ $\qquad x[n] = A^n x_0 + W_c \tilde{u}$

$$x[n] = A^n x_0 + W_c\tilde{u}$$

$$x[n] - A^n x_0 = W_c\tilde{u} \quad \Rightarrow \quad \underbrace{\tilde{u} = W_c^{-1}\left(x[n] - A^n x_0\right)}_{\text{we can go from } x_0 \text{ to } x_n \text{ by using input } \tilde{u} \text{ if } W_c^{-1} \text{ exists.}}$$

If W_c^{-1} is invertible, we can bring the system from x_0 to x_n using the input.

Observability

omit these to make it easier.

$$x[k+1] = Ax[k] + \left(Bu[k]\right)$$
$$y[k] = Cx[k] + \left(du[k]\right)$$

$$y[0] = Cx_0$$
$$y[1] = Cx[1] = CAx_0$$
$$y[2] = Cx[2] = CA^2x_0$$
$$\dots \quad y[n-1] = Cx[n-2] = CA^{n-1}x_0$$

$$y[n-1] = CA^{(n-1)} x_0$$

$$y = \begin{pmatrix} y[0] \\ y[1] \\ \vdots \\ y[n-1] \end{pmatrix} \qquad \overset{\text{observability matrix}}{W_o} = \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix}$$

$$y = W_o x_0$$

$$\Rightarrow x_0 = W_o^{-1} y$$

we can see the state x_0 from the output by multiplying it w/ W_o^{-1}!

Minimallity.

- Minimal if system is both controllable and observable.
- Non-minimal systems will have pole-zero cancellation in the transfer function
- Effectivelly low-order if minimal system.

Controllable Canonical Form.

Recall Canonical Form.

in state-space n^2+2n+1 parameters

in transfer-func $2n+1$ parameters ~~[illegible]~~ with

~~[illegible]~~ canonical form in state-space ~~[illegible]~~ $2n+1$ param

going from tf to ~~[illegible]~~ canonical form is easy

just plug in a's and b's.

<u>controllable canonical form.</u> $y[k] = (b_1 - a_1 b_0 \quad b_2 - a_2 b_0 \cdots b_n - a_n b_0)\, \bar{x}[k]$ [illegible]

$$\bar{x}[k+1] \left(\begin{array}{ccccc|c} -a_1 & -a_2 & \cdots & \cdots & -a_{n-1} & -a_n \\ \hline 1 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & 1 & \cdots & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & 1 & 0 \end{array}\right) \bar{x}[k] = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} u[k]$$

$$\bar{x}[k+1] = \left(\begin{array}{ccccc|c} -a_1 & -a_2 & \cdots & \cdots & -a_{n-1} & -a_n \\ 1 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & 1 & \cdots & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & 1 & 0 \end{array}\right) \bar{x}[k] + \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}}_{b} u[k]$$

$$\bar{W}_c = (\underbrace{\bar{b} \,|\, \bar{A}\bar{b}}_{} \,|\, \cdots \,|\, \bar{A}^{n-1}\bar{b})$$

selects first column of A as our b is just $\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

$$= \begin{pmatrix} 1 & -a_1 & a_1^2 - a_2 & -a_1^3 + 2a_1a_2 + a_3 & \cdots \\ 0 & 1 & -a_1 & a_1^2 - a_2 & \ddots \\ \vdots & 0 & 1 & -a_1 & \ddots \\ \vdots & \vdots & 0 & 1 & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \\ 0 & 0 & 0 & \cdots & \cdots 1 \end{pmatrix}$$

the '$-$' is to indicate this a system in controllable canonical form.

but this shit is whack ↗ first column is just b

(no bueno, so let's do something easier).

$A^2 b = A^* A b$

$= A^* \begin{pmatrix} -a_1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ $= -a_1 \times -a_1 + 1 \times -a_2 = a_1^2 - a_2$.

$$\bar{W}_c^{-1} = \begin{pmatrix} 1 & a_1 & a_2 & \cdots & a_{n-1} \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & a_2 \\ \vdots & & \ddots & \ddots & a_1 \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix}$$

$\bar{A} = TAT^{-1}$, $\bar{b} = Tb$, $\bar{c} = cT^{-1}$, $\bar{d} = d$ $\quad \bar{x}[k] = Tx[k]$.

→ But we need to find an appropriate T.

$$\bar{W}_c = (\bar{b} \quad \bar{A}\bar{b} \quad \bar{A}^2\bar{b} \quad \cdots \quad \bar{A}^{n-1}\bar{b})$$

this is in controllable canonical form

$$= (Tb \quad TAT^{-1}Tb \quad TAT^{-1}TAT^{-1}Tb \quad \cdots \quad TAT^{-1}TAT^{-1}\cdots TAT^{-1}Tb)$$

$$= (Tb \quad TAb \quad TA^2b \quad \cdots \quad TA^{n-1}b)$$

$$= TW_c$$

we know this.

hard to find anyways

$\bar{W}_c = TW_c \quad \Rightarrow \quad \bar{W}_c W_c^{-1} = T$

easy to find — easy peasy to find.

$$T^{-1} = W_c \bar{W}_c^{-1}$$

Ex.

$$x'(t) = \underbrace{\begin{pmatrix} 1 & 3 & -1 \\ 2 & -1 & -1 \\ 0 & 0 & 3 \end{pmatrix}}_{A} x(t) + \underbrace{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}}_{B} u(t)$$

$$T^{-1} = W_c \bar{W}_c^{-1}$$

AB ~~W_c~~ $A \cdot AB$

$$W_c = (B \;\; AB \;\; A^2B) = \begin{pmatrix} 1 & 2 & 14 \\ 0 & 3 & 4 \\ -1 & -3 & -9 \end{pmatrix}$$

$$\bar{W}_c^{-1} = \begin{pmatrix} 1 & a_1 & a_2 \\ 0 & 1 & a_1 \\ 0 & 0 & 1 \end{pmatrix}$$ find characteristic polynomial.

$$a(s) = \det(sI - A) = \det\begin{pmatrix} s-1 & -3 & 1 \\ -2 & s+1 & 1 \\ 0 & 0 & s-3 \end{pmatrix}$$

$$= (s-3) \det\begin{pmatrix} s-1 & -3 \\ -2 & s+1 \end{pmatrix}$$

$$= (s-3)\left[(s-1)(s+1) - (-2)(-3)\right]$$

$$= (s-3)(s^2-7)$$

$$= s^3 - 3s^2 - 7s + 21$$

$a_0 = 1, \; a_1 = -3, \; a_2 = -7, \; a_3 = 21$

$$\bar{W}_c^{-1} = \begin{pmatrix} 1 & -3 & -7 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix}$$

$$T^{-1} = W_c \bar{W}_c^{-1} = \begin{pmatrix} 1 & 2 & 14 \\ 0 & 3 & 4 \\ -1 & -3 & -9 \end{pmatrix}\begin{pmatrix} 1 & -3 & -7 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -1 & 1 \\ 0 & 3 & -5 \\ -1 & 0 & 7 \end{pmatrix} \qquad T = \begin{pmatrix} \frac{21}{19} & \frac{21}{57} & \frac{2}{19} \\ \frac{9}{19} & \frac{24}{57} & \frac{5}{19} \\ \frac{3}{19} & \frac{1}{19} & \frac{3}{19} \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 3 & -5 & 0 & 1 & 0 \\ -1 & 0 & 7 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & -2/3 & 1 & 1/3 & 0 \\ 0 & 1 & -5/3 & 0 & 1/3 & 0 \\ 0 & -1 & 8 & 1 & 0 & 1 \end{array}\right)$$

$\sim$