Exam **Mathematical Simulation**
Bachelor Knowledge Engineering, block 2.5
Thursday May 27$^{\text{th}}$, 2014, 09.00-12.00h
Please make sure that your answers are to-the-point and concise, but ensure that you give a **proper explanation**! *Ensure that your cell phone is turned off and is not within reach!* This is an open book exam. Please hand in the exam questions. The total number of points is 100

1. State whether the following propositions are true or false and argue why this is the case. Obviously the argument is very important.
   (a) (5 points) The bisection method can be used for the inverse transformation method without issue on distributions from the exponential family (e.g. Normal, exponential, Gamma, Erlang, etc.)
   (b) (5 points) When using the Box-Muller transform for generating standard normal random variates, the method takes two random numbers $u_1, u_2 \sim U(0,1)$ and returns $X_1 = \sqrt{-2\ln u_1}\cos(2\pi u_2)$ and $X_2 = \sqrt{-2\ln u_1}\sin(2\pi u_2)$ that are independent.
   (c) (5 points) Using the replication/deletion approach will ensure that you obtain normally distributed independent samples.
   (d) (5 points) When the null hypothesis of an empirical test is rejected the p-value indicates the variance of the random variable.
2. Two students, that are suppose to be attending the LP course of Steven and the MS course of Joël want to play a video game. Since the power outlets in 0.015 are not conveniently placed they have to rely on batteries for both of their laptops. Since Steven's 8.30h lecture and Joël's 11.00h are 4.5 hours of gaming, they stocked 7 laptop batteries. (Apparently the students ignore the inherent rudeness.) Initially the two students install batteries 1 and 2 respectively. If a battery fails the students pause the game, which along with the questions from the lecturers is a bit of distraction, and replace the depleted battery with the lowest numbered spare one. The lifetimes of the batteries are IID distributed according to an exponential distribution with average 1.2 hours. At a random time T a battery will be depleted and the stock will be empty. Only one battery, denoted X will not have failed
   (a) (10 points) What is P{X = i}?
   (b) (5 points) What is E[T]?
3. Consider the multiplicative LCG with multiplier $a = 6$, increment $c = 2$ modulus $m = 20$ and seed $Z_0 = 10$.
   (a) (5 points) Determine the period of this multiplicative LCG.
   (b) (5 points) Describe what happens if the seed is chosen to be $Z_0 = 1$.
4. (15 points) Consider the following pdf:

$$f(x) = \begin{cases} \frac{1}{8}x^2 & \text{if } -2 \leq x \leq 0 \\ x & \text{if } 0 < x \leq 1 \\ \frac{1}{x} & \text{if } 1 < x \leq e^{\frac{1}{6}} \\ 0 & \text{otherwise} \end{cases}$$

   Give the *inverse-transformation* algorithm for generating random variates from this distribution.
5. (10 points) Perform a Kolmogorov-Smirnov test ($\alpha = 0.05$) to determine whether you may reject the null hypothesis that these

$$0.0705, 0.7900, 0.8332, 0.8477, 0.3104, 0.1398, 0.8827, 0.6646, 0.7713, 0.3196, 0.7967, 0.4958,$$

$$0.7484, 0.3278, 0.0281, 0.2608, 0.5184, 0.6593, 0.6146, 0.1160$$

are random independent draws from the uniform (0, 1) distribution. What are your conclusions?

Case	**Adjusted test statistic**	**0.850**	**0.900**	**0.950**	**0.975**	**0.990**
All parameters known	$\left(\sqrt{n}+0.12+\frac{0.11}{\sqrt{n}}\right)D_n$	1.138	1.224	1.358	1.480	1.628
$\mathrm{N}\left(\bar{X}(n), S^2(n)\right)$	$\left(\sqrt{n}-0.01+\frac{0.85}{\sqrt{n}}\right)D_n$	0.775	0.819	0.895	0.955	1.035
$\mathrm{expo}\left(\bar{X}(n)\right)$	$\left(D_n-\frac{0.2}{n}\right)\left(\sqrt{n}+0.26+\frac{0.5}{\sqrt{n}}\right)$	0.926	0.990	1.094	1.190	1.308

6. A lecturer is interested in seeing whether the presence during the practicals improves the grade. To that end he recorded the average grade of two times ten groups of students, ten groups that did attend the practicals, ten groups that did not. The groups of students that did attend the practicals had average grades: 6.600, 9.130, 6.900, 9.630, 9.320, 9.820, 7.700, 9.790,9.750, 5.660, (sample variance 2.4602) and the groups that did not had average grades: 6.860, 7.040, 7.850, 7.740, 6.530, 5.610, 7.250, 5.250, 6.550, 7.340 (sample variance 0.7217).

   (a) (10 points) Use a t-test to test whether there is a significant difference between the two groups. Also provide the p-value.

   (b) (5 points) The lecturer came up with three methods to increase the presence during lectures. He measures the percentage of students that is present during 20 independent lectures for each approach. For approach one the mean presence is 67.8% with a standard deviation of 18.1, for approach two the average presence is 59.5% with standard deviation 14.8 and for approach three the average presence is 82.9% with standard deviation 13.1. How many additional replications are required to determine which approach is the best with 95% certainty; not caring about differences of 2%. The table for h_1 is provided below.

$\mathbf{P}^*$	n_0	$k=2$	$k=3$	$k=4$
0.90	20	1.896	2.342	2.583
0.90	40	1.852	2.283	2.514
0.95	20	2.453	2.872	3.101
0.95	40	2.386	2.786	3.003

7. (15 points) Consider the following Monte Carlo example where U represents a random number:

$$X_{1j} = U^5$$
$$X_{2j} = 1 + U^4$$

Analytically determine $\mathrm{Cov}(X_{1j}, X_{2j})$ and analytically calculate $\mathrm{Var}(X_{1j} - X_{2j})$ under both independent sampling and common random numbers. What are your conclusions? (once you arrive with your computations at the covariance and variance, you may use a decimal representation to avoid complicated fractions in the variances)

Success!